Mathematics 1120H – Calculus II: Integrals and Series

Trent University, Winter 2024

Solutions to Assignment #1
The Area of a Piece of a Parabola

Suppose a straight line is used to cut off a piece of a parabola. What is the area of this piece? The ancient Greek mathematician (and physicist and engineer) Archimedes of Syracuse (*c.* 287-212 B.C.) came up with a way to compute it† that is in some ways reminiscent of how Riemann sums work. The main differences are that it used triangles rather than rectangles and was not readily adaptable to compute the areas of other shapes for reasons which will probably become apparent if you work through this assignment. Please keep in mind, too, that in Archimedes' day, no one had any notion of symbolic algebraic notation or even a proper place-value number system, or of linking algebra to geometry via a coordinate system and functions, without most of which it's hard to imagine making Riemann sums work in general. It took most of two thousand years after Archimedes' time to invent all that ...

In this assignment, you will get to compute the area of a particular piece of a parabola in three different ways. First, the easy way:

1. Find the area of the finite region between the parabola $y = x^2$ and the line $y = 4$ by setting up and evaluating a suitable definite integral. The evaluation should be done both by hand and by using SageMath. *[2]*

Solution. The parabola $y = x^2$ and the line $y = 4$ intersect when $x^2 = 4$, *i.e.* when $x = \pm 2$. It's not hard to check that the finite region between the parabola and the line has $-2 \le x \le 2$ and that $x^2 \le 4$ for these values of x. It follows that the area of this region is given by the definite integral $\displaystyle\int_{-2}^{2} \left(4 - x^2\right)\, dx$. We proceed to evaluate this integral by hand and using SageMath:

By hand. The Power Rule will be our main tool:

$$\begin{aligned}\int_{-2}^{2} \left(4 - x^2\right)\, dx &= \left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} = \left(4 \cdot 2 - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right) \\ &= \left(8 - \frac{8}{3}\right) - \left(-8 - \frac{-8}{3}\right) = \frac{16}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3}\end{aligned}$$

Using SageMath.

```
[1]: integral(4-x^2,x,-2,2)

[1]: 32/3
```

Either way, we see that the area of the finite region between the parabola $y = x^2$ and the line $y = 4$ is $\dfrac{32}{3} = 10.666\dot{6}$. □

Second, a harder way, using a simplified version of Riemann integration:

2. Compute the definite integral you set up in solving **1** using the Right-Hand Rule formula, as described in the accompanying handout *Right-Hand Rule Riemann Sums*. The evaluation of the formula, once you've set it up, should be done both by hand and by using SageMath. *[4]*

† *Quadrature of the Parabola*, by Archimedes of Syracuse, translated by T.L. Heath, pp. 233-252 in *The Works of Archimedes*, Cambridge University Press, 1897. Reprinted by Dover Publications, ISBN-13: 9780486420844.

Note. You may the find the following summation formulas useful in doing **2**:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution. Per the handout, the general Right-Hand Rule formula is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left[\frac{b-a}{n} \cdot \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right)\right]$$

From the information about the region in the solution to **1** above, we have $a = -2$, $b = 2$, and $f(x) = 4 - x^2$, so in this case the Right-Hand Rule formula is

$$\begin{aligned}\int_{-2}^{2} \left(4 - x^2\right)\,dx &= \lim_{n\to\infty} \left[\frac{2-(-2)}{n} \cdot \sum_{i=1}^{n} f\left(-2 + i \cdot \frac{2-(-2)}{n}\right)\right] \\ &= \lim_{n\to\infty} \left[\frac{2-(-2)}{n} \cdot \sum_{i=1}^{n} \left[4 - \left(-2 + i \cdot \frac{2-(-2)}{n}\right)^2\right]\right],\end{aligned}$$

at least before we try simplifying and evaluating.

By hand. We will use a lot of algebra and the given summation formulas.

$$\begin{aligned}\int_{-2}^{2} \left(4 - x^2\right)\,dx &= \lim_{n\to\infty} \left[\frac{2-(-2)}{n} \cdot \sum_{i=1}^{n} \left[4 - \left(-2 + i \cdot \frac{2-(-2)}{n}\right)^2\right]\right] \\ &= \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left[4 - \left(-2 + \cdot \frac{4}{n} i\right)^2\right]\right] = \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left[4 - \left(4 - \frac{16}{n} i + \frac{16}{n^2} i^2\right)\right]\right] \\ &= \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left[\frac{16}{n} i - \frac{16}{n^2} i^2\right]\right] = \lim_{n\to\infty} \left[\frac{4}{n} \cdot \frac{16}{n} \cdot \sum_{i=1}^{n} \left[i - \frac{1}{n} i^2\right]\right] \\ &= \lim_{n\to\infty} \left[\frac{64}{n^2} \left[\left(\sum_{i=1}^{n} i\right) - \left(\frac{1}{n} \sum_{i=1}^{n} i^2\right)\right]\right] \\ &= \lim_{n\to\infty} \left[\frac{64}{n^2} \left[\frac{n(n+1)}{2} - \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}\right]\right] \\ &= \lim_{n\to\infty} \left[\frac{64}{n^2} \left[\frac{n(n+1)}{2} - \frac{(n+1)(2n+1)}{6}\right]\right] \\ &= \lim_{n\to\infty} \left[\frac{64}{n^2} \left[\frac{n^2+n}{2} - \frac{2n^2+3n+1}{6}\right]\right] = \lim_{n\to\infty} \left[\frac{64}{n^2} \left[\frac{3n^2+3n}{6} - \frac{2n^2+3n+1}{6}\right]\right] \\ &= \lim_{n\to\infty} \left[\frac{64}{n^2} \cdot \frac{n^2+1}{6}\right] = \lim_{n\to\infty} \left[\frac{64}{6} \left(\frac{n^2}{n^2} + \frac{1}{n^2}\right)\right] = \lim_{n\to\infty} \left[\frac{32}{3} \left(1 + \frac{1}{n^2}\right)\right] \\ &= \frac{32}{3}(1+0) = \frac{32}{3}\end{aligned}$$

Using SageMath. Here is a fairly general SageMath code fragment that could, in principle, be adapted to evaluating any definite integral using the Right-Hand Rule:

```
[2]: var("n")
     var("i")
     f = function('f')(x)
     f(x) = 4 - x^2
     a = -2
     b = 2
     s = function('s')(n)
     s(n) = sum( (b-a)/n * f(a+i*(b-a)/n), i, 1, n)
     limit( s(n), n=oo )

[2]: 32/3
```

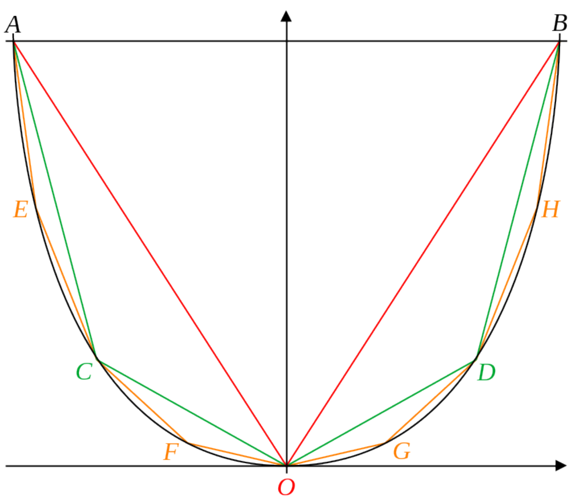
Again, either way, we see that the area of the finite region between the parabola $y = x^2$ and the line $y = 4$ is $\frac{32}{3}$. $\square$

Third, and hardest, is Archimedes' way, though we'll let him do the really heavy lifting. How did he do it? The key was the following result:

> THEOREM. Suppose a straight line intersects a parabola at points P and Q. Let R be the point on the part of the parabola between P and Q that is furthest from the line, and let S and T be the points on the parabola between P and R, and between Q and R, respectively, that are farthest from the line PR and the line QR, respectively. Then each of the triangles $\triangle PSR$ and $\triangle QTR$ has one-eighth of the area of the triangle $\triangle PQR$.

Proving this theorem is where Archimedes' did most of the really heavy lifting, but you can just use it for free. :-) One problem with using this result is that it is generally pretty difficult to find the initial point R on the part of the parabola between P and Q that is furthest from the line that cuts through the parabola from P to Q. Fortunately, this is not a problem in the particular setup we have on this assignment: the point on $y = x^2$ that is furthest from any horizontal line cutting through the parabola is the tip of the parabola, namely $(0, 0)$.

Here is a sketch of applying the process in Archimedes' theorem to the region described in question **1**, and then applying it again to each of the smaller parabolic segments created at the previous step:



0. Let A and B denote the points where the line $y = 4$ intersects the parabola $y = x^2$ and let O denote the origin $(0, 0)$, which is also the tip of the parabola. Connect each of A and B to O to form the triangle $\triangle AOB$.
1. Let C be the point on the arc of the parabola between A and O that is farthest from the line AO, and let D be the point on the arc of the parabola between B and O that is farthest from the line BO. Connect A and O to C to make triangle $\triangle ACO$, and connect B and O to D to make triangle $\triangle BDO$. By Archimedes' theorem, each of these triangles has one-eighth of the area of $\triangle AOB$.
2. Let E be the point on the arc of the parabola between A and C that is farthest from the line AC, let F be the point on the arc of the parabola between C and O that is farthest from the line CO, let G be the point on the arc of the parabola between D and O that is farthest from the line DO, and let H be the point on the arc of the parabola between B and D that is farthest from the line BD. Connect A and C to E to make triangle $\triangle AEC$, connect C and O to F to make triangle $\triangle CFO$, connect D and O to G to make triangle $\triangle DGO$, and connect B and D to H to make triangle BHD. By Archimedes' theorem each of these triangles has one-eighth of the area of each the triangles $\triangle ACO$ or $\triangle BDO$.

One could keep the process going through infinitely many steps. At each step you create twice as many triangles as you had at the previous step, and each of the new triangles has one-eighth the area of each of the triangles you had at the previous step.

Intuitively, it's easy to see that this process, if continued indefinitely, fills up the area of the parabolic region with these triangles. Also, so long as you know the area of the first triangle, $\triangle ABO$, which is pretty easy in this case, it's not hard to work out the collective areas of the triangles added at each step. There is no need, in particular, to keep working out what the furthest points from their base lines on various parabolic arcs actually are, which is usually pretty hard to do.

3. Use Archimedes' theorem and the observations above to find the area of the region described in question **1**. After you've set it up, you should do the computation both by hand and by using SageMath. *[4]*

Note. You may the find the following summation formula useful in doing **3**. So long as $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

The requirement that $|r| < 1$ is really necessary. Just for fun, try plugging in $a = 1$ and $r = 2$ to see what happens on each side ...

Solution. Consider the total area of all the triangles after each stage of the process described above.

0. At stage 0, we have $1 = 2^0$ triangles (base up and point down) with base 4 and height 4, and hence area $\frac{1}{2} \cdot 4 \cdot 4 = 8$.
1. At stage 1, we add $2 = 2^1$ triangles with area $8 \cdot \frac{1}{8}$, each, adding a combined area of $2 \cdot 8 \cdot \frac{1}{8} = 8 \cdot \frac{1}{4}$, for a total area of $8 + 8 \cdot \frac{1}{4}$.
2. At stage 2, we add $4 = 2^2$ triangles with area $8 \cdot \frac{1}{8} \cdot \frac{1}{8} = 8\left(\frac{1}{8}\right)^2$ each , adding a combined area of $2^2 \cdot 8\left(\frac{1}{8}\right)^2 = 8 \cdot \left(\frac{1}{4}\right)^2$, for a total area of $8 + 8 \cdot \frac{1}{4} + 8 \cdot \left(\frac{1}{4}\right)^2$.
3. At stage 2, we add $8 = 2^3$ triangles with area $8 \cdot \left(\frac{1}{4}\right)^2 \cdot \frac{1}{8}$ each, adding a combined area of $2^3 \cdot 8 \cdot \left(\frac{1}{4}\right)^3 = 8 \cdot \left(\frac{1}{4}\right)^3$, for a total area of $8 + 8 \cdot \frac{1}{4} + 8 \cdot \left(\frac{1}{4}\right)^2 + 8 \cdot \left(\frac{1}{4}\right)^3$.

⋮

Continuing the pattern developing above, we have:

n. At stage i, we add 2^i triangles with area $8 \cdot \left(\frac{1}{4}\right)^i$ each, adding a combined area of $2^i \cdot 8 \cdot \left(\frac{1}{4}\right)^i = 8 \cdot \left(\frac{1}{4}\right)^i$, for a total area of $8 + 8 \cdot \frac{1}{4} + 8 \cdot \left(\frac{1}{4}\right)^2 + 8 \cdot \left(\frac{1}{4}\right)^3 + \cdots + 8 \cdot \left(\frac{1}{4}\right)^i$.

Taking this to infinity and simplifying just a bit gives us the following series (*i.e.* infinite sum) that should add up to the area of the piece of the parabola described in question **1**:

$$\sum_{i=0}^{\infty} \frac{8}{4^i} = \frac{8}{4^0} + \frac{8}{4^1} + \frac{8}{4^2} + \frac{8}{4^3} + \frac{8}{4^4} + \cdots = 8 + 2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$$

By hand. The series $\displaystyle\sum_{i=0}^{\infty} \frac{8}{4^i} = 8 + 2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$ is a series fitting the pattern in the note above, with $a = 8$ and $r = \frac{1}{4}$. (Such a series is called a *geometric series* and r is called the *common ratio* of the series.) Since $\left|\frac{1}{4}\right| < 1$, we can apply the summation formula given in the note, yielding:

$$\sum_{i=0}^{\infty} \frac{8}{4^i} = 8 + 2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{8}{1 - \frac{1}{4}} = \frac{8}{\frac{3}{4}} = 8 \cdot \frac{4}{3} = \frac{32}{3}$$

Using SageMath. The slightly harder way, analogous to the code fragment used in the solution to question **2**:

```
[3]: t = function('t')(n)
     t(n) = sum( 8/4^i, i, 0, n )
     limit( t(n), n=oo )
```

```
[3]: 32/3
```

The easier way, using the fact that the sum command can handle infnite sums in many cases:

```
[4]: sum( 8/4^i, i, 0, oo )
```

```
[4]: 32/3
```

Once again, the area is $\frac{32}{3}$. □