Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2022

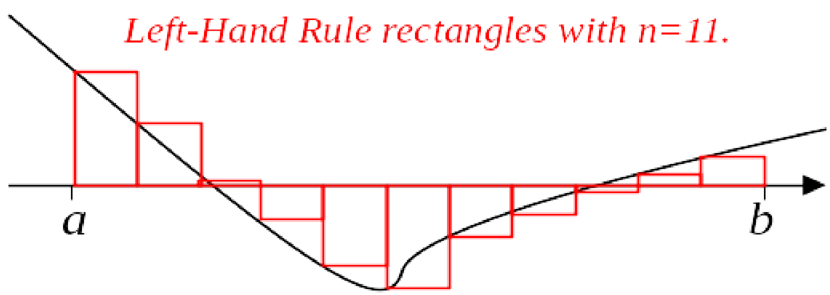
Assignment #1
Computing definite integrals with the Left-Hand Rule
Due on Friday, 21 January.
(May be submitted on paper or via Blackboard.)*

Recall that the definite integral $\int_a^b f(x)\,dx$ is the signed or weighted area of the region between $y = f(x)$ and the x-axis for $a \le x \le b$, where area above the x-axis is added and area below the x-axis is subtracted. It seems to be pretty hard to turn this idea into a complete and precise definition that can be used to prove all the basic properties of the definite integral, much less prove the Fundamental Theorem of Calculus, which relates the definite integral to computing antiderivatives. Indeed, many first-year calculus textbooks give highly simplified versions or even skip it entirely.† The definition given in §7.2 of our textbook is one of the more common highly simplified versions of this definition, often called the Left-Hand Rule. Please (at least!) skim through §7.2 before doing this assignment; for your convenience a summary of what you will need to know follows:

LEFT-HAND RULE. Suppose $f(x)$ is defined for all x in $[a, b]$ and is continuous at all but finitely many points of $[a, b]$. Then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{b-a}{n} f\left(a + (i-1)\cdot\frac{b-a}{n}\right)\right]$$

The idea is to divide up the interval $[a, b]$ into n subintervals of equal width $\triangle x = \frac{b-a}{n}$, so the ith subinterval, going from left to right and where $1 \le i \le n$, will be $\left[(i-1)\cdot\frac{b-a}{n}, i\cdot\frac{b-a}{n}\right]$. Each subinterval serves as the base of a rectangle of height $h_i = f\left(a + (i-1)\cdot\frac{b-a}{n}\right)$, which must then have area $\triangle x \cdot h_i = \frac{b-a}{n} f\left(a + (i-1)\cdot\frac{b-a}{n}\right)$.



* All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

† You can find the simplest version of a precise and complete definition of the definite integral that your instructor knows of in *A Precise Definition of the Definite Integral*, a handout which is among the supplementary materials in the *Course Content* section of the course Blackboard site.

It's called the Left-Hand Rule because it uses the left endpoint of each subinterval to evaluate $f(x)$ at to determine the height of the rectangle which has that subinterval as a base. Note that when the function dips below the x-axis, the rectangles have negative height and so the area formula gives negative areas.

The sum of the areas of these rectangles, the *nth Left-Hand Rule sum for* $\int_a^b f(x)\,dx$, namely $\sum_{i=1}^{n} \frac{b-a}{n} f\left(a+(i-1)\cdot\frac{b-a}{n}\right)$, approximates the area computed by $\int_a^b f(x)\,dx$. As we increase n and so shrink the width of the rectangles we get better and better approximations to the definite integral. The Left-Hand Rule will, in principle, properly compute $\int_a^b f(x)\,dx$ as long as $f(x)$ has at most finitely many removable or jump discontinuities and no vertical asymptotes in the interval $[a,b]$. Even some basic properties of definite integrals are hard to get if one were to try to use the Left-Hand Rule as the definition:

1. Suppose $f(x)$ is a function which is defined and continuous – and hence is integrable – on $\left[-1,\sqrt{2}\right]$. Explain why we would have a problem justifying

$$\int_{-1}^{0} f(x)\,dx + \int_{0}^{\sqrt{2}} f(x)\,dx = \int_{-1}^{\sqrt{2}} f(x)\,dx$$

if we used the Left-Hand Rule (or any rule that relies on subdividing $[a,b]$ into equal subintervals) as the actual definition of $\int_a^b f(x)\,dx$. *[2]*

Hint. It matters here that $\sqrt{2}$ is irrational.

2. Compute the nth Left-Hand Rule sum for $\int_0^4 \left(x^2+2x+3\right)\,dx$ for $n = 4$, 8, 16, and 32. *[2]*

Hint. Use mathematical software such as `SageMath`. (Unless you're a mathochist. :-)

3. Use the Left-Hand Rule to compute $\int_0^4 \left(x^2+2x+3\right)\,dx$ precisely. *[4]*

Hint. You'll need to do some algebra before taking the limit and may use the summation formulas $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ and $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

4. Compute $\int_0^4 \left(x^2+2x+3\right)\,dx$ precisely using antiderivatives. *[2]*