Lecture 20 Mar. 25th, 2022

Power Series : a series of the form $\sum_{n=0}^{\infty} c_n x^n$ where x is a variable.

• our desire is to rewrite functions of x as power series to make them easier to handle.

ex/

$\int e^{x^2} dx$ has no nice antiderivative, but you can write one as a power series.

Prototype : Geometric Series

$\frac{1}{1-x} = 1 + x + x^2 + \dots$ where $a = 1$, $r = x$ (ie. $\frac{a}{1-r}$)
which converges when $|r| = |x| < 1$

ex/ for which values of x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge ?

Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{x^{n+1}}{(n+1)!}\right)}{\left(\frac{x^n}{n!}\right)}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \text{ for all } x$$

So the series converges absolutely for all x.

This series sums to e^x. How can we tell?

$e^0 = 1$ and $\sum_{n=0}^{\infty} \frac{0^n}{n!} = \frac{0^0}{0!} + \cancel{\frac{0^1}{1!}}^{\,0} + \cancel{\frac{0^2}{2!}}^{\,0} + \dots = 1$

Is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$? Yes, they both satisfy the differential equation $\frac{dy}{dx} = y$ with the initial condition that $y = 1$ when $x = 0$,

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If $y = e^x$, then $\frac{dy}{dx} = \frac{d}{dx}e^x = e^x = y$, and $e^0 = 1$.

If $y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $\frac{dy}{dx} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$

$= 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = y$,

and $\sum_{n=0}^{\infty} \frac{0^n}{n!} = 1$.

The only way to explain this is that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Note: different functions could have the same power series, and so the power series is only equal to one of them.

ex/ $f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

has (via Taylor's formula) a power series expansion of:

$$0 + 0 + 0 + 0 + 0 + \ldots = \sum_{n=0}^{\infty} 0^{n+1}$$

which is the same as the expansion of $g(x) = 0$.

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ (when it converges) then

$f(0) = \sum_{n=0}^{\infty} c_n (0)^n = c_0(0)^0 + c_1(0)^1 + c_2(0)^2 + \ldots = c_0$

$f'(0) = \frac{d}{dx}\left(c_0 + c_1 x + c_2 x^2 + \ldots\right)\Big|_{x=0} = \left(0 + c_1 + 2c_2 x + 3c_3 x^2 + \ldots\right)\Big|_{x=0}$
$= c_1$

$f''(0) = \frac{d^2}{dx^2}\left(c_0 + c_1 x + c_2 x^2 + \ldots\right)\Big|_{x=0} = \frac{d}{dx}\left(c_1 + 2c_2 x + 3c_3 x^2 \ldots\right)\Big|_{x=0}$
$= \left(2c_2 + 6c_3 x + 12c_4 x^2 + \ldots\right)\Big|_{x=0} = 2c_2$

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$\overset{n^{th} \text{ derivative}}{}$

so $f^{(n)}(0) = n!\,c_n$

Taylor's formula:

If $f(x)$ is infinitely differentiable at $x=0$, then its Taylor Series is: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.