Series IV

Plan: finish 2.3, do 2.5 & some extra, go back & do 2.4

Does $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ converge or not?

Try the integral test

$$\int_1^{\infty} \frac{1}{x^{\sqrt{2}}}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^{\sqrt{2}}}\,dx$$

$$= \lim_{b\to\infty} \int_1^b x^{-\sqrt{2}}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-\sqrt{2}+1}}{-\sqrt{2}+1}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{b^{-\sqrt{2}+1}}{-\sqrt{2}+1} - \frac{1^{-\sqrt{2}+1}}{-\sqrt{2}+1}\right]$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-\sqrt{2}} \cdot \underbrace{\frac{1}{\underbrace{b^{\sqrt{2}-1}}_{\to\infty}}}_{=0} + \frac{1}{\sqrt{2}-1}\right) = \frac{1}{\sqrt{2}-1}$$

$\therefore$ the series converges

This worked (in the end) because $\sqrt{2} \approx 1.4\ldots > 1$

P-test:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$

Proof: throw the integral test at it now

Generalized p-test

$\sum_{n=1}^{\infty} \frac{a_k n^k + \ldots + a_1 n + a_0}{b_\ell n^\ell + \ldots + b_1 n + b_0}$ converges if $p = \ell - k > 1$ and diverges if $p = \ell - k \leq 1$

2.5 ~ (Basic Comparison test)

Suppose $\{a_n\}$ and $\{b_n\}$ are seqnences of positive terms. If $0 < a_n \leq b_n$ past some point, then...

(1) if $\sum_{n=0}^{\infty} b_n$ converge, so dose $\sum_{n=0}^{\infty} a_n$

(2) if $\sum_{n=0}^{\infty} a_n$ diverges, so dose $\sum_{n=0}^{\infty} b_n$

Ex: Does $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$ converge or not

Note that $\frac{1}{n^2+n+3} < \frac{1}{n^2}$

for $n \geq 1$ because $n^2+n+3 > n^2$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test [as $p=2>1$] it follows by the basic comparison test that $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$ converges too.

Ex: Does $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ converge?

$\frac{1}{n^2-n+3} < \frac{1}{n^2}$ but $n^2-n+3 < n^2$ for $n>3$

What do we compare $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ to if we want to show it converges?

$\frac{1}{n^2-n+3} < \frac{2}{n^2} = \frac{1}{n^2/2}$ past some point $\frac{n^2}{2} > n-3$ (namely $n \geq 3$)

but then $\underset{\frac{n^2}{2}}{\underset{\parallel}{n^2 - \frac{n^2}{2}}} < n^2-(n-3) = n^2-n+3$ so past $n \geq 3$,

we have $\frac{1}{n^2-n+3} < \frac{1}{n^2/2} = \frac{2}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges so does $\sum_{n=1}^{\infty} \frac{1}{n^2-n+3}$ by the comparison test

There's a better way: <u>The limit Comparison test</u>

Suppose $\{a_n\}$ & $\{b_n\}$ are (past some point) sequences of positive terms

Then, if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} d_n$ both diverges or both converges

[If $c=0$ then if $\sum_{n=0}^{\infty} a_n$ diverges so does $\sum_{n=0}^{\infty} b_n$, and if $\sum_{n=0}^{\infty} b_n$ converges so does $\sum_{n=0}^{\infty} a_n$]

[If $c=\infty$ then if $\sum_{n=0}^{\infty} a_n$ converges then so does $\sum_{n=0}^{\infty} b_n$, and if $\sum_{n=0}^{\infty} b_n$ diverges so does $\sum_{n=0}^{\infty} a_n$]

Ex: $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ look at the denominator terms in the numerator, 1, & the denominator, n^2 compare $\frac{1}{n^2-n+3}$ to $\frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-n+3}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1/n^2(n^2-n+3)}$$

$$= \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}+\frac{1}{n^2}}$$

$$= \frac{1}{1-0+0} = 1 > 0$$

So $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ converges (or not) exatly as $\sum_{n=0}^{\infty} \frac{1}{n^2}$ does but $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges by the p-test (since $p=2>1$) so $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ converges