Lecture 15 Mar 8^{th}, 2022

Recall: A sequence $\{a_n\}$ is a list of numbers indexed by the integers starting from some point. (ex/ $a_n = \frac{1}{n}$, $n \geq 1$)

A sequence $\{a_n\}$ converges as $n \to \infty$: for any $\epsilon > 0$ we can find an N such that for all $n \geq N$, $|a_n - L| < \epsilon$.

Useful trick: if $a_n = f(n)$ for all n and $f(x)$ is a function, and if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Series ($\sum_{n=0}^{\infty} a_n$): sum of sequence a_n if it exists

ex/ $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$

ex/ $\sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \ldots$

→ does not add up (except maybe ∞, which $\notin \mathbb{R}$)

A series converges (ie. adds up) to a sum $A \in \mathbb{R}$ if it's sequence of partial sums converges to A.
The partial sum up to n of the series is:

$a_0 + a_1 + a_2 + \ldots + a_n = S_n$

So $\sum_{n=0}^{\infty} a_n$ converges to A if $\lim_{n\to\infty} S_n = A$.

ex/ $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = \dfrac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}$

$1 + \frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{n^2} = ?$

Prototypes for Summing Series:

1) Geometric Series

$\sum_{n=0}^{\infty} ar^n$ where a is first term, $r_1 = \frac{ar^{n+1}}{ar^n}$

$= a + ar + ar^2 + \ldots + ar^{n-1} + ar^n$

$= a(1 + r + r^2 + \ldots + r^{n-1} + r^n)$

If we multiply this by $(1-r)$, we get

$\Rightarrow a(1 + r + r^2 + \ldots + r^{n-1} + r^n)(1-r)$

$= a(1 + r + r^2 + \ldots + r^{n-1} + r^n - r - r^2 - \ldots - r^{n-1} - r^n - r^{n+1})$

$= a(1 - r^{n+1})$

$\Rightarrow a(1 + r + r^2 + \ldots + r^{n-1} + r^n) = a\frac{1-r^{n+1}}{1-r}$

So $\sum_{n=0}^{\infty} ar^n$ has a limit if $\lim_{n\to\infty} a\frac{1-r^{n+1}}{1-r}$ exists.

It converges exactly when either $a = 0$ or when $|r| < 1 \Rightarrow -1 < r < 1$.

If $|r| < 1$, then $r^{n+1} \to 0$ so the limit is $\frac{a}{1-r}$.

If $r > 1$, then $r^{n+1} \to \infty$.

If $r = 1$, we'd be dividing by 0.

If $r = -1$, then $r^{n+1} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$

If $r < -1$, then r^{n+1} oscillates and there is no limit.

2) Telescoping Series

ex/ $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$

$= 1\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{4} + \ldots$

$$S_n = \sum_{i=0}^{n} \frac{1}{(i+1)(i+2)} = 1\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{4} + \ldots + \frac{1}{(n+1)(n+2)}$$

$$\frac{1}{(i+1)(i+2)} = \frac{A}{i+1} + \frac{B}{i+2}$$

$$= \frac{A(i+2) + B(i+1)}{(i+1)(i+2)}$$

$$= \frac{(A+B)i + (2A+B)}{(i+1)(i+2)}$$

$\Rightarrow 1 = (A+B)i + (2A+B)$

$\Rightarrow A+B=0$ and $2A+B=1$

$\Downarrow$

$B=-A \Rightarrow 2A+(-A)=1 \Rightarrow \underline{A=1} \Rightarrow B=-1$

so $\displaystyle\frac{1}{(i+1)(i+2)} = \frac{1}{i+1} - \frac{1}{i+2}$

Thus $1\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{4} + \ldots + \frac{1}{(n+1)(n+2)}$

$= \left(1-\cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right) + \cancel{\ldots} + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)$

$= 1 - \frac{1}{n+2}$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right) = 1-0 = 1.$$

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=0}^{\infty} a_n$ diverges (ie. does not converge).

Why? If $a_n \to L \neq 0$ then $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots + L + L + \ldots$

$\Rightarrow$ if $L>0$, $\sum_{n=0}^{\infty} a_n = \infty$; if $L<0$, $\sum_{n=0}^{\infty} a_n = -\infty$

If the individual 'a_n's bounce around to make $\lim_{n\to\infty} a_n$ fail, then the partial sums bounce around too.

Altenative: Suppose $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} S_n = L$.

ie. for any $\epsilon>0$ there is an N such that if $n \geq N$ then $|S_n - L| < \frac{\epsilon}{2}$.

$$
\begin{aligned}
\text{Then } |a_n| &= |S_n - S_{n-1}| \\
&= |S_n - L + L - S_{n-1}| \\
&\leq |S_n - L| + |L - S_{n-1}| \\
&= |S_n - L| + |S_{n-1} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

Thus for any $\epsilon>0$, there is an N such that if $n \geq N$ then $|a_n - 0| = |a_n| < \epsilon$.
ie. $\lim_{n\to\infty} a_n = 0$.

Unfortunately, the Divergence test is of limited use, there are lots of series for which $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=0}^{\infty} a_n$ diverges.

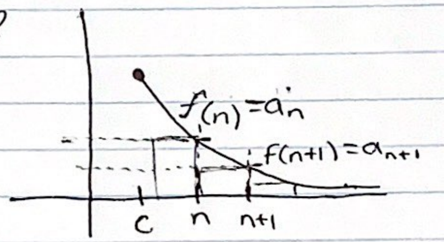
ex/ $\sum_{n=0}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ survives divergence test, but the series adds up to ∞, and ∞ isn't a number, so the series diverges.

Integral Test

Suppose $\sum_{n=c}^{\infty} a_n$ comes from a function $f(x)$ via $a_n = f(n) \geq 0$.

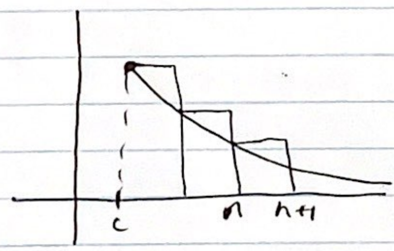
If $f(x)$ is a decreasing function and integrable on $[c, \infty)$, then $\sum_{n=c}^{\infty} a_n$ converges or diverges exactly as the improper integral $\int_c^{\infty} f(x)\,dx$ does.

Why?



Area of all the rectangles $= \sum_{n=c}^{\infty} a_n \leq \int_c^{\infty} f(x)\,dx$

If the integrals exists, the sum does to.



This gives $\sum_{n=c}^{\infty} a_n \geq \int_c^{\infty} f(x)\,dx$.

ex/ $\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \int_1^{\infty} \frac{1}{x}\,dx = \lim_{c\to\infty} \int_1^{c} \frac{1}{x}\,dx = \lim_{c\to\infty} \ln(x)\Big|_1^c$
$= \lim_{c\to\infty} (\ln(c) - \cancel{\ln(1)}^{0}) = \lim_{c\to\infty} \ln(c) = \infty.$

$\therefore$ the series diverges.

ex/ $\sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow \int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{c\to\infty} \int_1^{c} x^{-2}\,dx = \lim_{c\to\infty} \frac{-1}{x}\Big|_1^c$
$= \lim_{c\to\infty} \left(-\frac{1}{c} - \frac{-1}{1}\right) = \lim_{c\to\infty} (1 - \cancel{\tfrac{1}{c}}^{0}) = 1.$

$\therefore$ the series converges (but we don't know what it actually adds up to).