Lecture 10

Feb 11th, 2022

Proper Definite Integral: $\int_a^b f(x)dx$, $a,b \in \mathbb{R}$

Improper: $\int_{-\infty}^{\infty} xe^{-x^2} dx$, $\int_0^{\infty} xe^{-x^2} dx$

- ∞ cannot be thrown around like a number

ex/ $\int_0^{\infty} xe^{-x^2} dx = \lim_{a\to\infty} \int_0^a xe^{-x^2} dx$

$\boxed{\text{Let } w = -x^2 \Rightarrow dw = -2x\,dx \Rightarrow -\frac{1}{2}dw = x\,dx}$

$= \lim_{a\to\infty} \int_{x=0}^{x=a} e^{w}\left(-\frac{1}{2}dw\right) = \lim_{a\to\infty} -\frac{1}{2}\int_{x=0}^{x=a} e^{w} dw$

$= \lim_{a\to\infty} -\frac{1}{2}e^{w}\Big|_{x=0}^{x=a} = \lim_{a\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^a$

$= \lim_{a\to\infty}\left[-\frac{1}{2}e^{-a^2} - \left(-\frac{1}{2}e^{0^2}\right)\right] = \lim_{a\to\infty}\left[-\frac{1}{2}e^{-a^2} + \frac{1}{2}\right]$

$= \lim_{a\to\infty} \frac{1}{2}\left(1 - e^{-a^2}\right) = \frac{1}{2}(1-0) = \frac{1}{2}$ ∎

ex/ $\int_0^1 \frac{1}{\sqrt{x}} dx$ → asymptote as $x \to 0^+$

$= \lim_{b\to 0^+} \int_b^1 \frac{1}{\sqrt{x}} dx = \lim_{b\to 0^+} \int_b^1 x^{-1/2} dx$

$= \lim_{b\to 0^+} \frac{x^{-1/2+1}}{-1/2+1}\Big|_b^1 = \lim_{b\to 0^+} 2\sqrt{x}\Big|_b^1$

$= \lim_{b\to 0^+} (2\sqrt{1} - 2\sqrt{b}) = \lim_{b\to 0^+} (2 - 2\sqrt{b})$

$= 2 - 0 = 2$ ∎

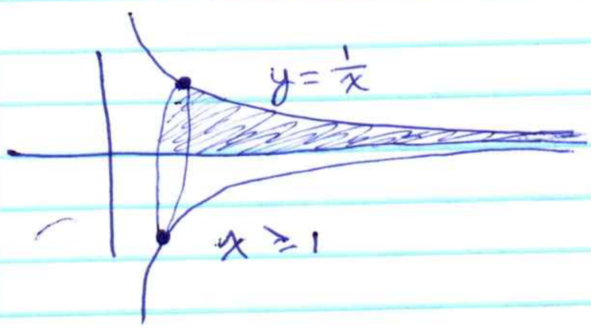
ex/ $\int_{-1}^{1} \ln(|x|)\,dx = \int_{-1}^{0} \ln(|x|)dx + \int_{0}^{1} \ln(|x|)\,dx$

$= \lim_{a\to 0^-} \int_{-1}^{a} \ln(|x|)dx + \lim_{b\to 0^+} \int_{b}^{1} \ln(|x|)\,dx$

$= \lim_{a\to 0^-} \int_{-1}^{a} \ln(-x)dx + \lim_{b\to 0^+} \int_{b}^{1} \ln(x)dx$

* then solve using dummy product parts.

What is the volume of "Gabriel's Trumpet"?



• revolve region below $y=\frac{1}{x}$ and above $y=0$ about x-axis ($x \geq 1$)

$V = \int_{1}^{\infty} \pi\left(\frac{1}{x}\right)^2 dx = \lim_{a\to\infty} \int_{1}^{a} \frac{\pi}{x^2}\,dx$

$= \lim_{a\to\infty} \int_{1}^{a} \pi x^{-2}\,dx = \lim_{a\to\infty} \frac{\pi x^{-1}}{-1}\Big|_{1}^{a}$

$= \lim_{a\to\infty} -\frac{\pi}{x}\Big|_{1}^{a} = \lim_{a\to\infty} \left[-\frac{\pi}{a} - \left(-\frac{\pi}{1}\right)\right]$

$= \lim_{a\to\infty} \left(-\frac{\pi}{a} + \pi\right) = 0 + \pi = \pi$ $\square$