Trigonometric Substitutions

wherein we substitute the variable with a trig function

eg $\int \sqrt{1-x^2}\, dx$ Substitute $x = \sin(\theta)$ $\Rightarrow \theta = \arcsin(x)$, so $dx = \cos(\theta)\, d\theta$,
to take advantage of $\boxed{1-\sin^2(x) = \cos^2(x)}$.

$$= \int \sqrt{1-\underbrace{\sin^2(\theta)}_{x^2}} \cdot \underbrace{\cos(\theta)\, d\theta}_{dx} = \int \sqrt{\cos^2(\theta)} \cdot \cos(\theta)\, d\theta$$

$$= \int \cos(\theta) \cdot \cos(\theta)\, d\theta = \int \cos^2(\theta)\, d\theta \qquad \boxed{\cos^2(\theta) = \tfrac{1}{2}(1+\cos(2\theta))}$$

$$= \int \tfrac{1}{2}(1+\cos(2\theta))\, d\theta = \tfrac{1}{2}\int (1+\cos(w)) \cdot \tfrac{1}{2}\, dw = \tfrac{1}{4}(w+\sin(w)) + C$$

$w = 2\theta$
$dw = 2d\theta \Rightarrow d\theta = \frac{1}{2}dw$

$$\boxed{\sin(2\theta) = 2\sin(\theta)\cos(\theta)}$$

$$= \tfrac{1}{4}(2\theta + \sin(2\theta)) + C = \tfrac{1}{4}(2\theta + 2\sin(\theta)\cos(\theta)) + C$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{2}\sin(\theta)\cos(\theta) + C = \boxed{\tfrac{1}{2}\arcsin(x) + \tfrac{1}{2}x\sqrt{1-x^2} + C}$$

Moral: see $\sqrt{1-x^2}$ in the integrand, consider $x = \sin(\theta)$,
especially if nothing easier is obvious.

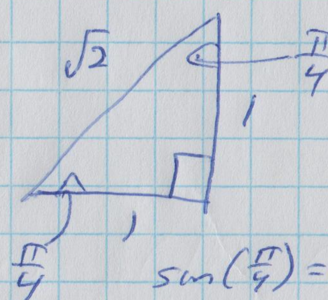
es $\int_0^1 \sqrt{1+x^2}\, dx$

Substitute $x = \tan(t)$, so $dx = \sec^2(t)\, dt$, to take advantage of $1+\tan^2(t) = \sec^2(t)$.

Change limits:

x	t
0	0
1	$\frac{\pi}{4}$

$\tan(t) = 1$
$= \sin(t)/\cos(t)$
$\Rightarrow \sin(t) = \cos(t)$



$\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\Downarrow$

$\sec(\frac{\pi}{4}) = \sqrt{2}$

$$= \int_0^{\pi/4} \sqrt{1+\underbrace{\tan^2(t)}_{x^2}} \cdot \underbrace{\sec^2(t)\, dt}_{dx}$$

$$= \int_0^{\pi/4} \sqrt{\sec^2(t)} \cdot \sec^2(t)\, dt$$

$$= \int_0^{\pi/4} \sec(t) \sec^2(t)\, dt = \int_0^{\pi/4} \sec^3(t)\, dt$$

(Use the reduction formula.)

$$= \frac{1}{3-1} \tan(t) \sec^{3-2}(t) \Big|_0^{\pi/4} - \frac{3-2}{3-1} \int_0^{\pi/4} \sec^{3-2}(t)\, dt$$

$$= \frac{1}{2} \tan(t) \sec(t) \Big|_0^{\pi/4} - \frac{1}{2} \int_0^{\pi/4} \sec(t)\, dt$$

Moral: see $\sqrt{1+x^2}$ in the integrand consider $x = \tan(t)$.

$$= \left(\frac{1}{2}\tan(\tfrac{\pi}{4})\sec(\tfrac{\pi}{4}) - \frac{1}{2}\tan(0)\sec(0)\right) - \frac{1}{2} \ln(\sec(t)+\tan(t)) \Big|^{\pi/4}$$

$$= \frac{1}{2}\cdot 1 \cdot \sqrt{2} - \frac{1}{2}\cdot 0 \cdot 1 - \left[\frac{1}{2}\ln(\overset{\sqrt{2}}{\sec(\tfrac{\pi}{4})} + \overset{1}{\tan(\tfrac{\pi}{4})}) - \frac{1}{2}\ln(\overset{1}{\sec(0)} + \overset{0}{\tan(0)})\right]$$

$$= \left(\frac{\sqrt{2}}{2} - 0\right) - \left[\frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{2}\overset{0}{\ln(1+0)}\right] = \boxed{\frac{1}{\sqrt{2}} - \frac{1}{2}\ln(1+\sqrt{2})}$$

eg $$\int_1^{\sqrt{2}} \frac{1}{\sqrt{x^2-1}}\,dx$$

Substitute $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)d\theta$, (3)
to take advantage of $\sec^2(\theta) = 1 + \tan^2(\theta)$.

Change limits:

x	θ
1	0
$\sqrt{2}$	$\frac{\pi}{4}$

$\sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1$

$$= \int_0^{\pi/4} \frac{1}{\sqrt{\underbrace{\sec^2(\theta)}_{x^2}-1}} \cdot \underbrace{\sec(\theta)\tan(\theta)\,d\theta}_{dx} = \int_0^{\pi/4} \frac{1}{\sqrt{\tan^2(\theta)}} \sec(\theta)\tan(\theta)\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{\cancel{\tan(\theta)}} \sec(\theta)\cancel{\tan(\theta)}\,d\theta = \int_0^{\pi/4} \sec(\theta)\,d\theta$$

$$= \ln\left(\sec(\theta) + \tan(\theta)\right)\Big|_0^{\pi/4} = \ln\left(\sec\left(\tfrac{\pi}{4}\right) + \tan\left(\tfrac{\pi}{4}\right)\right) - \ln(\sec(0) + \tan(0))$$

$$= \ln(\sqrt{2}+1) - \ln(1+0) = \boxed{\ln(\sqrt{2}+1)} - \cancel{\ln(1)}^{\,0}$$

Moral: see $\sqrt{x^2-1}$ in the integrand, consider substituting $x = \sec(\theta)$.

Ex $$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

Try $u = 1-x^2$, so $du = (-2x)dx$

$\Rightarrow x\,dx = \left(-\frac{1}{2}\right)du$, $x = \sqrt{1-u}$.

This gives $$\int \frac{\overbrace{\sqrt{1-u}}^{x}}{\underbrace{\sqrt{u}}_{\sqrt{1-x^2}}} \underbrace{\left(-\frac{1}{2}\right)du}_{x\,dx}$$

... which is not promising.

$x = \sin(t) \Rightarrow t = \arcsin(x)$
$\qquad\qquad\qquad\quad = \sin^{-1}(x)$

$dx = \cos(t)\,dt$

$$= \int \frac{\sin^2(t)}{\sqrt{1-\sin^2(t)}} \cos(t)\,dt$$

$$= \int \frac{\sin^2(t)\cos(t)}{\sqrt{\cos^2(t)}}\,dt = \int \frac{\sin^2(t)\cancel{\cos(t)}}{\cancel{\cos(t)}}\,dt$$

Moral: $\sqrt{1-x^2}$ substitute $x = \sin(t)$ ..

$$= -\frac{1}{2}\sin^{2-1}(t)\cos(t) + \frac{2-1}{2}\int \underbrace{\sin^{2-2}(t)}_{=\sin^0(t) = 1}\,dt$$

$$= -\frac{1}{2}\sin(t)\cos(t) + \frac{1}{2}\int 1\,dt = -\frac{1}{2}\sin(t)\cos(t) + \frac{1}{2}t + C$$

$$= -\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(t) + C$$

Next time: More complicated quadratics inside the root.