Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Solutions to Quiz #5
Tuesday, 23 February.

Do *both* of the following questions. Show all your work! Simplify where you conveniently can.

1. Find the domain and all of the vertical and horizontal asymptotes, if any, of

$$f(x) = \ln\left(\frac{\pi}{2} - \arctan(x)\right). \text{ [2]}$$

SOLUTION. $\ln(t)$ is defined for all $t > 0$ and $\arctan(x)$ is defined for all x, and we also have $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$ for all x. It follows that $\frac{\pi}{2} - \arctan(x) > 0$ for all x, so $f(x) = \ln\left(\frac{\pi}{2} - \arctan(x)\right)$ is defined for all x, *i.e.* the domain of $f(x)$ is $(-\infty, \infty) = \mathbb{R}$.

Since $f(x) = \ln\left(\frac{\pi}{2} - \arctan(x)\right)$ is a composition of continuous functions, it is continuous everywhere it is defined. As it is defined everywhere, it follows that it cannot have any vertical asymptotes.

It remains to check for horizontal asymptotes. Recall that $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$ for all x. As $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}$, so $\frac{\pi}{2} - \arctan(x) \to \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi^-$, and as $x \to +\infty$, $\arctan(x) \to \frac{\pi}{2}$, so $\frac{\pi}{2} - \arctan(x) \to \frac{\pi}{2} - \frac{\pi}{2} = 0^+$. It follows that:

$$\lim_{x\to-\infty} \ln\left(\frac{\pi}{2} - \arctan(x)\right) = \lim_{t\to\pi^-} \ln(t) = \ln(\pi) \approx 1.1447$$
$$\lim_{x\to+\infty} \ln\left(\frac{\pi}{2} - \arctan(x)\right) = \lim_{t\to 0^+} \ln(t) = -\infty$$

Thus $f(x)$ has a horizontal asymptote of $y = \pi$ as $x \to -\infty$, but no horizontal asymptote in the other direction. ■

2. Find the domain and all of the vertical and horizontal asymptotes, if any, of

$$g(x) = \frac{\ln(x) - \ln(x+1)}{\ln(x) - \ln(x-1)}. \text{ [3]}$$

SOLUTION. $\ln(t)$ is defined and 1–1 for all $t > 0$. Since it is 1–1, the denominator of $g(x)$, $\ln(x) - \ln(x-1)$, will never be 0 when $\ln(x)$ and $\ln(x-1)$ are both defined. $g(x) = \frac{\ln(x) - \ln(x+1)}{\ln(x) - \ln(x-1)}$ will therefore be defined exactly when all of $\ln(x)$, $\ln(x)$, and $\ln(x-1)$ are defined, *i.e.* exactly when all of $x > 0$, $x+1 > 0$, and $x - 1 > 0$ are true. This happens exactly when $x > 1$, so the domain of $g(x)$ is $(1, \infty) = \{\, x \mid x > 1 \,\}$.

Since $g(x) = \frac{\ln(x) - \ln(x+1)}{\ln(x) - \ln(x-1)}$ is a composition of continuous functions, it is continuous wherever it is defined. This means it cannot have any vertical asymptotes except the point where the definition starts to fail, namely $x = 1$. We check to see what happens at this point.

$g(x)$ is only defined to the right of $x = 1$, so we only have to (or can!) check on that side. As $x \to 1^+$, we have $\ln(x) \to \ln(1) = 0^+$, $\ln(x+1) \to \ln(1+1) = \ln(2)$, and $\ln(x-1) \to -\infty$ (as $x - 1 \to 0^+$). Thus

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \frac{\ln(x) - \ln(x+1) \to 0 - \ln(2)}{\ln(x) - \ln(x-1) \to 0 - (-\infty)} = -\infty\,,$$

so $g(x)$ has a vertical asymptote on the right side of $x = 1$, headed downwards.

It remains to check for horizontal asymptotes. Since $g(x)$ is only defined for $x > 1$, we only have to check what happens as $x \to +\infty$. Note that as $x \to +\infty$, $\ln(x)$, $\ln(x)$, and $\ln(x-1)$ all head to $+\infty$, too. What about the differences of these in the numerator and denominator?

$$\begin{aligned}
\lim_{x \to +\infty} (\ln(x) - \ln(x+1)) &= \lim_{x \to +\infty} \ln\left(\frac{x}{x+1}\right) = \lim_{x \to +\infty} \ln\left(\frac{x}{x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}\right) \\
&= \lim_{x \to +\infty} \ln\left(\frac{1}{1 + \frac{1}{x}}\right) = \ln\left(\frac{1}{1+0}\right) = \ln(1) = 0 \\
\lim_{x \to +\infty} (\ln(x) - \ln(x-1)) &= \lim_{x \to +\infty} \ln\left(\frac{x}{x-1}\right) = \lim_{x \to +\infty} \ln\left(\frac{x}{x-1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}\right) \\
&= \lim_{x \to +\infty} \ln\left(\frac{1}{1 - \frac{1}{x}}\right) = \ln\left(\frac{1}{1-0}\right) = \ln(1) = 0
\end{aligned}$$

This means that the limit that we are really interested in, $\lim\limits_{x \to +\infty} g(x)$ is an indeterminate ratio to which we can apply l'Hôpital's Rule:

$$\begin{aligned}
\lim_{x \to +\infty} g(x) &= \lim_{x \to +\infty} \frac{\ln(x) - \ln(x+1) \to 0}{\ln(x) - \ln(x-1) \to 0} = \lim_{x \to +\infty} \frac{\frac{d}{dx}\left[\ln(x) - \ln(x+1)\right]}{\frac{d}{dx}\left[\ln(x) - \ln(x-1)\right]} \\
&= \lim_{x \to +\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{\frac{1}{x} - \frac{1}{x-1}} = \lim_{x \to +\infty} \frac{\frac{x+1-x}{x(x+1)}}{\frac{x-1-x}{x(x-1)}} = \lim_{x \to +\infty} \frac{\frac{1}{x(x+1)}}{\frac{-1}{x(x-1)}} \\
&= \lim_{x \to +\infty} \frac{1}{x(x+1)} \cdot \frac{x(x-1)}{-1} = \lim_{x \to +\infty} \frac{-(x-1)}{x+1} = \lim_{x \to +\infty} \frac{1-x}{1+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x \to +\infty} \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1} = \frac{0-1}{0+1} = -1
\end{aligned}$$

Thus $g(x)$ has a horizontal asymptote of $y = -1$ as $x \to +\infty$. For those who care, it approaches this asymptote from above. (Why?) ■

[Total = 5]