<u>A fairly tedious example</u> $\qquad f(x) = \frac{1}{1-x} = (1-x)^{-1}$

To apply Taylor's formula, we need to figure out how the n^{th} derivatives of $f(x)$ work out:

$$f'(x) = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}\cdot \frac{d}{dx}(1-x) = \cancel{(-1)}(1-x)^{-2}\cancel{(-1)}$$
$$= 1\cdot(1-x)^{-2}$$

$$f''(x) = \frac{d}{dx}(1-x)^{-2} = (-2)(1-x)^{-3}\frac{d}{dx}(1-x) = (-2)(1-x)^{-3}(-1)$$
$$= 2(1-x)^{-3} = 1\cdot 2\cdot(1-x)^{-3}$$

$$f'''(x) = \frac{d}{dx}2(1-x)^{-3} = 2(-3)(1-x)^{-4}\frac{d}{dx}(1-x) = (-1)2\cdot 3(1-x)^{-4}(-1)$$
$$= 1\cdot 2\cdot 3\cdot(1-x)^{-4} = 6(1-x)^{-4} = 3!\,(1-x)^{-4}$$

$$f^{(4)}(x) = \frac{d}{dx}3!\,(1-x)^{-4} = 3!\cdot(-4)(1-x)^{-5}\frac{d}{dx}(1-x)$$
$$= 4\cdot 3!\cdot(1-x)^{-5} = 4!\,(1-x)^{-5}$$

... and so on. In general, if $n \geq 1$, then

$$f^{(n)}(x) = n!\,(1-x)^{-(n+1)}, \quad \text{and so}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	
0	$(1-x)^{-1}$	1	$[=0!]$
1	$1!\cdot(1-x)^{-2}$	$1!$	
2	$2!\cdot(1-x)^{-3}$	$2!$	
3	$3!\cdot(1-x)^{-4}$	$3!$	
$\vdots$	$\vdots$	$\vdots$	

In particular, we have that $f^{(n)}(0) = n!$ for all $n \geq 0$.

It follows that the Taylor series of $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ (at 0) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = \sum_{n=0}^{\infty} \frac{n!\,x^n}{n!} = \sum_{n=0}^{\infty} x^n$$

$$= 1 + x + x^2 + x^3 + \cdots .$$

Of course, we already knew from the formula for the sum of a geometric series that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad \left[\begin{array}{l}\text{First term } a=1\\ \text{\& common ratio } r=x.\end{array}\right]$$

which converges exactly when $|x|<1$. This brings us to

Dirty Trick #2: If you have (or can find) a power series equal to $f(x)$, that power series is the Taylor series of $f(x)$.

Here's an example of this, starting with $\frac{1}{1-x} = 1 + x + x^2 + \cdots$.

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = \frac{1}{1-x}\cdot\frac{1}{1-x} = (1+x+x^2+x^3+\cdots)(1+x+x^2+x^3+\cdots)$$

$$= 1\cdot(1+x+x^2+x^3+\cdots) + x\cdot(1+x+x^2+x^3+\cdots) + x^2\cdot(1+x+x^2+\cdots) + \cdots$$

$$\begin{aligned} = {} & 1 + x + x^2 + x^3 + \cdots \\ & + x + x^2 + x^3 + \cdots \\ & + x^2 + x^3 + \cdots \\ & \qquad \vdots \end{aligned} \qquad = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$$

Thus, with a bit of algebra, we get that the Taylor series (at 0) of $f(x) = (1-x)^{-2}$ is $1 + 2x + 3x^2 + \cdots$.

(Of course, we could also have used the fact that $\frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}$ and Dirty Trick #1 to do this.)

A less spectacular, but more common, example of Dirty Trick #2 would be the following:

Suppose you want the Taylor series at 0 of $f(x) = xe^x$, and you already know that the Taylor series of e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$. Then the Taylor series of $f(x)$ is just

$$x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots .$$

<u>Another tedious example</u> $\quad f(x) = \ln(1+x)$

Cutting to the chase

n	$f^{(n)}(x)$	$f^{(n)}(0)$	
0	$\ln(1+x)$	0	$[= \ln(1+0)]$
1	$\frac{1}{1+x} = (1+x)^{-1}$	1	
2	$(-1)(1+x)^{-2}$	-1	
3	$1\cdot 2\,(1+x)^{-3}$	2	
4	$-1\cdot 2\cdot 3\,(1+x)^{-4}$	-6	
$\vdots$	$\vdots$	$\vdots$	
k	$(-1)^{k+1}(k-1)!(1+x)^{-k}$	$(-1)^{k+1}(k-1)!$	
$\vdots$	$\vdots$	$\vdots$	

so Taylor's formula tells us that the Taylor series of $f(x) = \ln(1+x)$ at 0 is

$$0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!\,x^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

A faster way to get this would be to observe that

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C$$

$$\shortparallel$$

$$\int (1-x+x^2-x^3+\cdots)\,dx \qquad \left(\text{since } \frac{1}{1+x} = \frac{1}{1-(-x)} = 1-x+x^2-x^3+\cdots\right)$$

$$\shortparallel$$

$$\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + C'.$$

Since plugging in $x=0$ makes both $\ln(1+x)$ & the series $=0$, we can conclude that $C=C'$, and so

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

This brings us to:

<u>Dirty Trick #3</u>: (really, just Dirty Trick #1 in reverse)

If you know the Taylor series of $f'(x)$ and integrate it term-by-term, then you get the Taylor series of $f(x)$, up to a constant.

... and you can usually pin the constant down by plugging in a suitable value of x, as was done in the example above.

This brings us to the end of the course.

(While we don't have the time to get into it, I would urge those interested in how to use partial sums of Taylor series to approximate functions to look at Section 11.11, and, in particular, at Theorem 11.11.1. This theorem is due to Lagrange, not Taylor, so the section is slightly misnamed.)

Thank you all for bearing with us in these all too interesting times. I hope to see you all again once things settle down.

Yours,

[signature]

P.S.: One last little useful fact. The Taylor series of the derivative or integral of $f(x)$ will have the same radius of convergence as the Taylor series of $f(x)$. However, they might behave differently at the endpoints of the interval of convergence, and so have (slightly) different intervals of convergence.