Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to Assignment #5.1 – the `Maple`-less edition
Guaranteeing Convergence

INSTRUCTIONS: *You may do* one *of Assignment #5.1 and the original Assignment #5. Either way, please submit your solutions on or by the due date using the assignment submission tool on Blackboard, preferably as a pdf. If that doesn't work, please email it to your instructor.*

It's a fact that $\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots$. (We'll see why this series adds up to $\frac{1}{e}$ once we do Taylor series.)

1. Find a value of m such that $\sum_{n=0}^{k} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \cdots + \frac{(-1)^k}{k!}$ is guaranteed to be within $0.0001 = 10^{-4}$ of $\frac{1}{e}$ for all $k \geq m$ and explain why it's guaranteed. *[3]*

SOLUTION. We will use the fact that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ satisfies the conditions given in the Alternating Series Test.

First, since $n! \geq 1 > 0$ for all $n \geq 0$ and $(-1)^n$ alternates sign as n increases because $(-1)^{n+1} = -(-1)^n$, the terms of the series, $\frac{(-1)^n}{n!}$, alternate sign.

Second, because we have $(n+1)! = (n+1) \cdot n! \geq n!$ for all $n \geq 0$, it follows that $\left| \frac{(-1)^{n+1}}{(n+1)!} \right| = \frac{1}{(n+1)!} \leq \frac{1}{n!} = \left| \frac{(-1)^n}{n!} \right|$ for all $n \geq 0$.

Third, $\lim_{n \to \infty} \left| \frac{(-1)^n}{n!} \right| = \lim_{n \to \infty} \frac{1 \to 1}{n! \to \infty} = 0$.

It is a consequence of the discussion on page 272 in Section 11.4 of the textbook, and made explicit on page 2 of the lecture notes of 2020-03-18, that if $\sum_{n=0}^{\infty} a_n$ is a series meeting the conditions of the Alternating Series Test and A is the sum of the series, then the kth partial sum $\sum_{n=0}^{k} a_n$ of the series is within $|a_{k+1}|$ of A, *i.e.* $\left| \left(\sum_{n=0}^{\infty} a_n \right) - A \right| \leq |a_{k+1}|$. In our situation it means that it is guaranteed that

$$\left| \left(\sum_{n=0}^{k} \frac{(-1)^n}{n!} \right) - \frac{1}{e} \right| \leq \left| \frac{(-1)^{k+1}}{(k+1)!} \right| = \frac{1}{(k+1)!}\,.$$

Thus, if we ensure that $\frac{1}{(k+1)!} < 0.0001$, we will also ensure that $sum_{n=0}^{k} \frac{(-1)^n}{n!}$ is within 0.0001 of $\frac{1}{e}$.

How big does k have to be to make $\frac{1}{(k+1)!} < 0.0001 = \frac{1}{10000}$? We will get this inequality exactly when $(k+1)! > 10000$. Since $7! = 5040 < 10000$ and $8! = 40320 > 10000$, we need to have $k+1 \geq 8$, *i.e.* $k \geq 7$. Note that because $n!$ is strictly increasing for $n \geq 1$, $\frac{1}{n!}$ is strictly decreasing for $n \geq 1$. Thus, once we have a k such that $\frac{1}{(k+1)!} < 0.0001 = \frac{1}{10000}$, every larger of value of k will also give $\frac{1}{(k+1)!} < 0.0001 = \frac{1}{10000}$.

Thus if $k \geq 7$, it is guaranteed that

$$\left| \left(\sum_{n=0}^{k} \frac{(-1)^n}{n!} \right) - \frac{1}{e} \right| \leq \left| \frac{(-1)^{k+1}}{(k+1)!} \right| = \frac{1}{(k+1)!} \leq \frac{1}{(7+1)!} < 0.0001\,.$$

Hence $m = 7$ does the job. ■

It's also a fact that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$. (Again, we'll see why this series adds up to $\frac{\pi}{4}$ once we do Taylor series.)

2. Find a value of m such that $\sum_{n=0}^{k} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^k}{2k+1}$ is guaranteed to be within $0.0001 = 10^{-4}$ of $\frac{\pi}{4}$ for all $k \geq m$ and explain why it's guaranteed. *[3]*

SOLUTION. We will use the fact that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ satisfies the conditions given in the Alternating Series Test.

First, since $2n+1 \geq 1 > 0$ for all $n \geq 0$ and $(-1)^n$ alternates sign as n increases, the terms of the series, $\frac{(-1)^n}{2n+1}$, alternate sign.

Second, because we have $2(n+1)+1 = 2n+3 > 2n+1$ for all $n \geq 0$, it follows that $\left| \frac{(-1)^{n+1}}{2(n+1)+1} \right| = \frac{1}{2n+3} < \frac{1}{2n+1} = \left| \frac{(-1)^n}{2n+1} \right|$ for all $n \geq 0$.

Third, $\lim_{n \to \infty} \left| \frac{(-1)^n}{2n+1} \right| = \lim_{n \to \infty} \frac{1}{2n+1} \begin{matrix} \to 1 \\ \to \infty \end{matrix} = 0$.

Similarly to the solution to **1** above, it follows that

$$\left| \left(\sum_{n=0}^{k} \frac{(-1)^n}{2n+1} \right) - \frac{\pi}{4} \right| \leq \left| \frac{(-1)^{k+1}}{2(k+1)+1} \right| = \frac{1}{2k+3}\,.$$

Thus, if we ensure that $\frac{1}{2k+3} < 0.0001$, we will also ensure that $\sum_{n=0}^{k} \frac{(-1)^n}{2n+1}$ is within 0.0001 of $\frac{\pi}{4}$.

How big does k have to be to ensure that $\frac{1}{2k+3} < 0.0001 = \frac{1}{10000}$? Well, we need to have $2k+3 > 10000$, *i.e.* $k > \frac{10000-3}{2} = \frac{9997}{2} = 4998.5$. The least such k is 4999. Similarly to the argument in the solution to **1**, since $2k+3$ is strictly increasing with k, it follows that once we a k that works, every larger k will as well.

Thus if $k \geq 4999$, its is guaranteed that

$$\left| \left(\sum_{n=0}^{k} \frac{(-1)^n}{2n+1} \right) - \frac{\pi}{4} \right| \leq \left| \frac{(-1)^{k+1}}{2(k+1)+1} \right| = \frac{1}{2k+3} \leq \frac{1}{2 \cdot 4999 + 3} < 0.0001 \,.$$

Hence $m = 4999$ does the job. ■

The series $\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}$ also adds up to $\frac{\pi}{4}$. (To see why, do a little algebra to answer question **3**.)

3. How are these two series adding up to $\frac{\pi}{4}$ related, besides having the same sum? *[4]*

SOLUTION. The terms in this series, $\frac{2}{(4n+1)(4n+3)}$, look a lot like the kind of things we used partial fractions simplify in order to integrate them. Perhaps simplifying them in the same way here would help. Applying the partial fraction technology:

$$\begin{aligned} \frac{2}{(4n+1)(4n+3)} &= \frac{A}{4n+1} + \frac{B}{4n+3} \\ &= \frac{A(4n+3)+B(4n+1)}{(4n+1)(4n+3)} = \frac{(4A+4B)n + (3A+B)}{(4n+1)(4n+3)} \end{aligned}$$

It follows that $4A+4B=0$ and $3A+B=2$. Solving for B in the second equation, $B = 2-3A$, and substituting into the first equation gives $4A+4(2-3A)=0$, so $4A+8-12A=0$, and thus $-8A = 0-8 = -8$. It follows that $A=1$ and $B = 2-3\cdot 1 = -1$. Thus

$$\frac{2}{(4n+1)(4n+3)} = \frac{1}{4n+1} + \frac{-1}{4n+3}$$

What happens when we plug this into the given series?

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)} &= \sum_{n=0}^{\infty} \left(\frac{1}{4n+1} + \frac{-1}{4n+3} \right) \\ &= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \left(\frac{1}{9} - \frac{1}{11} \right) + \cdots \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \end{aligned}$$

The two series are basically the same series, allowing for a little algebraic manipulation. The "combined term" form has some advantages in practice, though. It is a series of positive terms, so it converges absolutely, where the other only converges conditionally, and it turns you need a lot fewer terms to get a given degree of precision out of the partial sums. ■