Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2019

Solutions to Assignment #5
Serious Stuff

One can learn in class (or read in the textbook, or look it up elsewhere) that the *harmonic series*,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots ,$$

diverges, that is, doesn't add up to a real number. (Technically, this means that the limit of the partial sums, $\lim_{k \to \infty} \left[\sum_{n=1}^{k} \frac{1}{n} \right]$ doesn't exist.) On the other hand, the *alternating harmonic series*,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots ,$$

converges, that is, it does add up to a real number. (Technically, this means that the limit of the partial sums, $\lim_{k \to \infty} \left[\sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \right]$, exists. The sum of the series is, by definition, the limit of the partial sums.) This is usually shown using the Alternating Series Test (see §11.4 in the textbook). Your first task will be to see whether a couple of other modifications of the harmonic series converge or not.

1. Does the series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$, in which every third number in the harmonic series gets subtracted instead of added, converge or diverge? *[3]*

SOLUTION. We will think of this series in terms of groups of three terms, which lets us write it a bit more compactly as $\sum_{k=0}^{\infty} \left[\frac{1}{3k+1} - \frac{1}{3k+2} + \frac{1}{3k+3} \right]$. We will combine each group of three into a single term, which will put in a form we can more conveniently apply a suitable convergence test to. Observe that

$$\begin{aligned} \frac{1}{3k+1} - \frac{1}{3k+2} + \frac{1}{3k+3} &= \frac{(3k+2)(3k+3) - (3k+1)(3k+3) + (3k+1)(3k+2)}{(3k+1)(3k+2)(3k+3)} \\ &= \frac{9k^2 + 15k + 6 - 9k^2 - 12k - 3 + 9k^2 + 9k + 2}{27k^3 + 54k^2 + 33k + 6} \\ &= \frac{9k^2 + 12k + 5}{27k^3 + 54k^2 + 33k + 6} , \end{aligned}$$

so $\sum_{k=0}^{\infty} \left[\frac{1}{3k+1} - \frac{1}{3k+2} + \frac{1}{3k+3} \right] = \sum_{k=0}^{\infty} \frac{9k^2 + 12k + 5}{27k^3 + 54k^2 + 33k + 6}$. Consider the latter form: since $p = 3 - 2 = 1 \not> 1$ for this series, it follows by the Generalized p-Test that the series diverges. □

2. Does the series $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \cdots$, in which the last two of every group of three numbers in the harminic series get subtracted instead of added, converge or diverge? *[3]*

SOLUTION. We will take the same approach we did in the solution to **1** and think of the series in terms of groups of three terms, writing it as $\sum_{k=0}^{\infty} \left[\frac{1}{3k+1} - \frac{1}{3k+2} - \frac{1}{3k+3} \right]$. Once again, we will combine each group of three into a single term, which will put in a form we can more conveniently apply a suitable convergence test to. Observe that

$$\begin{aligned}
\frac{1}{3k+1} - \frac{1}{3k+2} - \frac{1}{3k+3} &= \frac{(3k+2)(3k+3) - (3k+1)(3k+3) - (3k+1)(3k+2)}{(3k+1)(3k+2)(3k+3)} \\
&= \frac{9k^2 + 15k + 6 - 9k^2 - 12k - 3 - 9k^2 - 9k - 2}{27k^3 + 54k^2 + 33k + 6} \\
&= \frac{-9k^2 - 6k + 1}{27k^3 + 54k^2 + 33k + 6},
\end{aligned}$$

so $\sum_{k=0}^{\infty} \left[\frac{1}{3k+1} - \frac{1}{3k+2} - \frac{1}{3k+3} \right] = \sum_{k=0}^{\infty} \frac{-9k^2 - 6k + 1}{27k^3 + 54k^2 + 33k + 6}$. Consider the latter form: since $p = 3 - 2 = 1 \not> 1$ for this series, it follows by the Generalized p-Test that the series diverges. □

NOTE. The harmonic series and its variants are a very rich source of examples of why series can be tricky. The following movie poster, modified by some University of Toronto engineers, kind of makes this point ... :-)

Your second task is to do a bit of algebra with power series, which are basically like polynomials of infinite degree.

3. Suppose x is a variable and a_n for $n \geq 0$ are constants such that

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= \left(1 + x + x^2 + x^3 + \cdots\right)^2 = \left(\sum_{n=0}^{\infty} x^n\right)^2 .\end{aligned}$$

Find a formula for a_n in terms of n. *[4]*

Hint: Work out the first few a_ns by multiplying out $\left(1 + x + x^2 + x^3 + \cdots\right)^2$ and then collecting like terms, and look for a pattern.

Solution. Let's follow the hint:

$$\begin{aligned}\left(1 + x + x^2 + x^3 + \cdots\right)^2 &= \left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right) \\ &= \quad 1\left(1 + x + x^2 + x^3 + \cdots\right) \\ &\quad + x\left(1 + x + x^2 + x^3 + \cdots\right) \\ &\quad + x^2\left(1 + x + x^2 + x^3 + \cdots\right) \\ &\quad + x^3\left(1 + x + x^2 + x^3 + \cdots\right) \\ &\quad \vdots \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ &\qquad + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ &\qquad\quad + x^2 + x^3 + x^4 + x^5 + \cdots \\ &\qquad\qquad + x^3 + x^4 + x^5 + \cdots \\ &\qquad\qquad\qquad \ddots \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\end{aligned}$$

The coefficient of x^n in the final series is $n + 1$, *i.e.* $a_n = n + 1$ for $n \geq 0$. ■