Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2018

Solutions to Assignment #2
The Gamma Function

One of the big uses of integrals in various parts of mathematics is to define functions that are otherwise difficult to nail down. For example, consider the factorial function on the non-negative integer, defined by $0! = 1$ and $(n+1)! = n! \cdot (n+1)$. (It's pretty easy to check that if $n \geq 1$ is an integer, then $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$.) The factorial function turns up in many parts of mathematics, including algebra, calculus [wait till we do series!], combinatorics, and number theory. The essentially discrete factorial function has a continuous (also differential and integrable) counterpart, which also comes up a fair bit in both applied and theoretical mathematics, namely the gamma function $\Gamma(x)$. This can be defined in a number of different ways, but the easiest definition to work with is in terms of an integral:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{a\to\infty}\int_0^a t^{x-1}e^{-t}\,dt$$

This definition makes sense for all $x > 0$.

1. Verify that $\Gamma(1) = 1$. *[3]*

SOLUTION. We'll be using the substitution $u = -t$, so $du = (-1)\,dt$ and $dt = (-1)\,du$.

$$\begin{aligned}\Gamma(1) &= \lim_{a\to\infty}\int_0^a t^{1-1}e^{-t}\,dt = \lim_{a\to\infty}\int_0^a t^0 e^{-t}\,dt = \lim_{a\to\infty}\int_0^a e^{-t}\,dt = \lim_{a\to\infty}\int_{x=0}^{x=a} e^u\,(-1)\,du \\ &= \lim_{a\to\infty} (-1)e^u\Big|_{x=0}^{x=a} = \lim_{a\to\infty} (-1)e^{-x}\Big|_0^a = \lim_{a\to\infty}\left[(-1)e^{-a} - (-1)e^{-0}\right] \\ &= \lim_{a\to\infty}\left[-\frac{1}{e^a} + 1\right] = -0 + 1 = 1 \quad \text{since } e^a \to \infty \text{ as } a \to \infty. \quad \blacksquare\end{aligned}$$

2. Show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. *[3]*

SOLUTION. We will first compute the definite integral $\displaystyle\int_0^a t^x e^{-t}\,dt$, where $x > 0$, using integration by parts with $u = t^x$ and $v' = e^{-t}$, so $u' = xt^{x-1}$ and $v = (-1)e^{-t}$. Note that x is just some constant as far as the variable in the integral, namely t, is concerned, and see the solution to **1** above for the antiderivative of e^{-t}.)

$$\begin{aligned}\int_0^a t^x e^{-t}\,dt &= t^x e^{-t}\Big|_0^a - \int_0^a xt^{x-1}(-1)e^{-t}\,dt = \left(a^x e^{-a} - 0^x e^{-0}\right) - (-1)x\int_0^a t^{x-1}e^{-t}\,dt \\ &= a^x e^{-a} + x\int_0^a t^{x-1}e^{-t}\,dt \quad (\text{Since } 0^x = 0 \text{ if } x > 0.)\end{aligned}$$

We plug this into the definition of $\Gamma(x+1)$ and chug away. Suppose $x>0$:

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^{(x+1)-1}e^{-t}\,dt = \lim_{a\to\infty}\int_0^a t^{(x+1)-1}e^{-t}\,dt = \lim_{a\to\infty}\int_0^a t^{x}e^{-t}\,dt \\
&= \lim_{a\to\infty}\left[a^x e^{-a} + x\int_0^a t^{x-1}e^{-t}\,dt\right] = \left[\lim_{a\to\infty} a^x e^{-a}\right] + \left[\lim_{a\to\infty} x\int_0^a t^{x-1}e^{-t}\,dt\right] \\
&= 0 + x\lim_{a\to\infty}\int_0^a t^{x-1}e^{-t}\,dt \quad \text{(Since } e^{-a}\to 0 \text{ faster than } a^x\to\infty \text{ as } a\to\infty.\text{)} \\
&= x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x) \quad \blacksquare
\end{aligned}$$

3. Use **1** and **2** to show that $\Gamma(n+1)=n!$ for every integer $n\geq 0$. *[2]*

Solution. Recall that the factorial function is defined for all integers $n\geq 0$ by $0!=1$ and $(n+1)! = n!\cdot(n+1)$.

First, by the solution to **1**, we have that $\Gamma(0+1)=\Gamma(1)=1=0!$, which takes care of $n=0$.

Second, suppose that we have verified that $\Gamma(k+1)=k!$ for some particular integer $k\geq 0$. Then, by the solution to **2**, we have that $\Gamma\left((k+1)+1\right) = (k+1)\Gamma(k+1) = (k+1)\cdot k! = (k+1)!$.

It follows from these two facts that $\Gamma(n+1)=n!$ for every integer $n\geq 0$:

$$\begin{aligned}
&\Gamma(0+1) = 0! \quad \text{by the first fact} \\
\Longrightarrow\ &\Gamma(1+1) = \Gamma\left((0+1)+1\right) = (0+1)! = 1! \quad \text{by the second fact} \\
\Longrightarrow\ &\Gamma(2+1) = \Gamma\left((1+1)+1\right) = (1+1)! = 2! \quad \text{by the second fact} \\
\Longrightarrow\ &\Gamma(3+1) = \Gamma\left((2+1)+1\right) = (2+1)! = 3! \quad \text{by the second fact} \\
\vdots\ \ &\ \vdots
\end{aligned}$$

The argument above is technically a proof by induction: the first fact is the base step of the proof, and the second fact is the inductive step of the proof. $\blacksquare$

4. What is $\Gamma\left(\frac{1}{2}\right)$? *[2]*

Solution. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Why? Well, one could just look it up, but let's go whole hog and do it. Well, not quite whole hog – we'll suppress the limits which are technically part of evaluating improper integrals and toss ∞ around like it was a number.

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{\frac{1}{2}-1}e^{-t}\,dt = \int_0^\infty t^{-1/2}e^{-t}\,dt = \int_0^\infty \frac{1}{\sqrt{t}}\cdot e^{-t}\,dt \\
&= \int_0^\infty e^{-u^2}\,2\,du \quad \begin{array}{l}\text{Using the substitution } u=\sqrt{t},\\ \text{so } du = \frac{1}{2\sqrt{t}}\,dt \text{ and } \frac{1}{\sqrt{t}}\,dt = 2\,du.\end{array} \quad \text{Also: } \begin{array}{ccc} x & 0 & \infty \\ u & 0 & \infty \end{array} \\
&= 2\int_0^\infty e^{-u^2}\,du = \int_{-\infty}^\infty e^{-u^2}\,du \quad \text{Since } e^{-u^2} = e^{-(-u)^2} \text{ for all } u.
\end{aligned}$$

It remains to show that $\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-u^2}\,du$ works out to $\sqrt{\pi}$. It is actually easier to show that $\left(\int_{-\infty}^{\infty} e^{-u^2}\,du\right)^2 = \pi$, but this does require a bit of multivariate calculus and the use of polar coordinates.

Without getting into the technicalities, two key facts that will be exploited below are that in multivariate calculus we usually try to handle things one variable at a time and that a function of one independent variable is simply a constant as far as another independent variable is concerned. We will also use the fact that if we do the change of coordinate systems in the plane from Cartesian coordinates to polar coordinates via $x = r\cos(\theta)$ and $y = r\sin(\theta)$, then $dx\,dy$ gets replaced by $r\,dr, d\theta$. (Feel free to look it up or ask about it.)

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-u^2}\,du\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,r\,dr\,d\theta \qquad \begin{array}{l}\text{Substitute } w = -r^2\text{, so } dw = -2r\,dr \text{ and} \\ r\,dr = -\frac{1}{2}\,dw\text{, with } \begin{array}{ccc} r & 0 & \infty \\ w & 0 & -\infty \end{array}.\end{array} \\
&= \int_{0}^{2\pi}\int_{0}^{-\infty} e^{w}\left(-\frac{1}{2}\right)dw\,d\theta = \int_{0}^{2\pi}\frac{1}{2}\int_{-\infty}^{0} e^{w}\,dw\,d\theta \\
&= \frac{1}{2}\int_{0}^{2\pi}\left[e^{w}\big|_{-\infty}^{0}\right]d\theta = \frac{1}{2}\int_{0}^{2\pi}\left[e^{0} - e^{-\infty}\right]d\theta = \frac{1}{2}\int_{0}^{2\pi}[1-0]\,d\theta \\
&= \frac{1}{2}\int_{0}^{2\pi} 1\,d\theta = \frac{1}{2}\theta\bigg|_{0}^{2\pi} = \frac{1}{2}\cdot 2\pi - \frac{1}{2}\cdot 0 = \pi
\end{aligned}$$

It follows that $\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$. ■