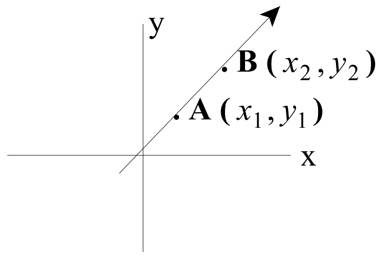
Lines, Conics, Tangents, Limits and the Derivative

The Straight Line

Any two points on the (x,y) plane when joined form a line segment. If the line segment is extended beyond the two points then it is called a straight line. The points A (x_1, y_1) and B (x_2, y_2) represent the two general points on any line.



The slope "m" of the line containing the points A and B is given by: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

The length of the line segment AB or $|AB|$ is given by: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Equations of Lines

Parallel lines have the same slope and perpendicular lines have slopes that are negative reciprocals of each other. ie. 3 and $\frac{-1}{3}$.

Lines that are parallel to the x axis have a zero slope (ie. $\frac{0}{\mathbf{k}}$, $\mathbf{k} \in \mathbf{R}$) and are given in the form y = b where "b" is the y intercept.

Lines that are parallel to the y axis have a slope that is undefined (ie. $\frac{\mathbf{k}}{\mathbf{0}}$, $\mathbf{k} \in \mathbf{R}$) and are given in the form x = a where "a" is the x intercept.

To find the equation of a particular line you must be given or be able to find a point on that line and its slope. Each line is unique and there will be only one equation with the given characteristics. Any of three forms for the equation can be used.

A) Slope y- intercept

y = mx + b	m = slope of the line b = y intercept

B) Standard Form

ax + by + c = 0	a, b, and c are integers and a > 0 $m = \frac{-a}{b}$

C) Slope Form

$\dfrac{(y - y_1)}{(x - x_1)} = m$ or	m = slope of the line
$(y - y_1) = m (x - x_1)$	(x_1, y_1) is a point on the line.

Examples:

1. Given the points A (-3,4) and B (6, 3) find the slope of AB and the length of AB.

m AB $= \frac{y_2 - y_1}{x_2 - x_1}$	Use the formula for slope
$= \frac{3-4}{6-(-3)}$	Substitute in the values for the x and y coordinates.
$= \frac{-1}{9}$	Simplify.
$\lvert AB \rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Use the formula for length or distance.
$= \sqrt{(9)^2 + (-1)^2}$	Substitute in the values for the x and y coordinates.
$= \sqrt{82}$	Simplify.

2. Find the equation of the line passing through (-2, 1) with a slope of $\frac{1}{2}$.

$\frac{y-1}{x-(-2)} = \frac{1}{2}$	Use slope form for the equation of the line.
2(y - 1) = 1 (x + 2)	Multiply both sides by the denominators.
2y - 2 = x + 2	Simplify.
x - 2y + 4 = 0	Rearrange in standard form.

3. Find the equations of the lines through (-3, 5) and
   a) parallel to y = 2x - 7
   b) perpendicular to y = 2x - 7

a) m = 2	The given line has slope 2 as y = <u>2</u>x - 7
∴ The required line is	
y = 2x + b	Use slope y intercept form
5 = 2(-3) + b	Substitute in the values of x and y.
b = 11	Solve for b.
∴ y = 2x + 11	Substitute in the values for m and b.

b) m = $\frac{-1}{2}$	Use the negative reciprocal for perpendicular slope.
∴ $\frac{y-5}{x-(-3)} = \frac{-1}{2}$	Substitute values in the slope form.
2(y - 5) = -1(x + 3)	Multiply both sides by the denominators.
2y - 10 = -x - 3	Simplify.
x + 2y - 7 = 0	Write in standard form.

4. Find the equation of the line through (2, - 8)
   a) parallel to x axis

b) perpendicular to x axis

a) $y = b$ — Zero slope

$y = -8$ — Substitute in the y value to find b.

b) $x = a$ — This is the equation for parallel to y axis or perpendicular

$x = 2$ — to x axis. Substitute in the x value of the given point.

X and Y intercepts

The y - intercept (y_i) is the point where the line crosses the y axis. It is found by letting x = 0 in the equation of the line and solving for y. The x - intercept (x_i) is the point where the line crosses the x axis. It is found by letting y = 0 and solving for x. Points that are on a line satisfy the equation of the line.

Examples:

1. Determine whether the given points are on the line $3x - 2y + 1 = 0$.

a) (3, 5)

$3x - 2y + 1$	Use the left side of the equation.
$= 3(3) - 2(5) + 1$	Substitute in the values for x and y
$= 9 - 10 + 1$	Evaluate.
$= 0$	Left side = Right side

$\therefore$ (3, 5) is on the given line.

b) (-1, 2)

$3x - 2y + 1$	
$= -3 + 4 + 1$	
$= 2$	Left side $\neq$ Right side

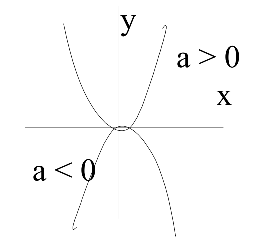
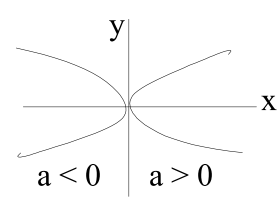
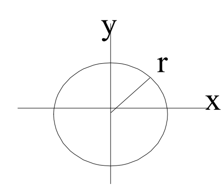
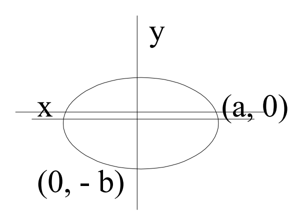
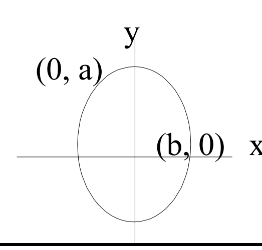
$\therefore$ (-1, 2) is not on the line.

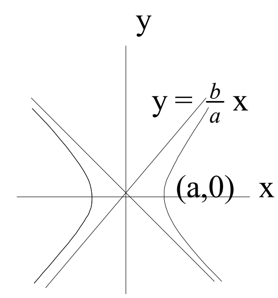
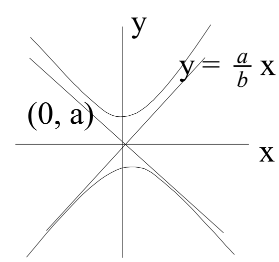
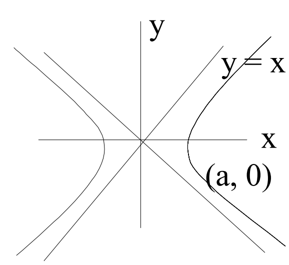
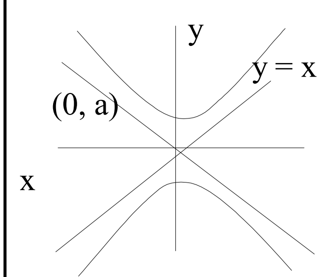
2. Find the x and y intercepts for the line $3x - 2y + 12 = 0$

y_i let $x = 0$	
$-2y + 12 = 0$	Solve for y.
$y = 6$	
x_i let $y = 0$	
$3x + 12 = 0$	Solve for x.
$x = -4$	

Conic Sections

The curves on the x,y plane are called conics. Their equations show the position and shape of the conic. Notice the differences and similarities of all the conic section equations. When solving a problem choose the most appropriate equation and find the values for the missing variables. Each can be identified by their equation. Look carefully at the plus and minus signs as well as where the terms are placed.

Name	Equation	Description	Sketch
Parabola	$y = ax^2$	- vertex (0,0) - $a > 0$ opens up - $a < 0$ opens down - symmetric about y axis	
Parabola	$x = a y^2$	- vertex (0, 0) - inverse of $y = a x^2$ - $a > 0$ opens right - $a < 0$ opens left - symmetric about x axis	
Circle	$x^2 + y^2 = r^2$	- centre (0, 0) - radius = r	
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a > b$	- centre (0, 0) -major axis (longest) on x axis = 2a - minor axis on y axis = 2b - vertices at ($\pm$ a, 0)	
Ellipse	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $a > b$	- centre (0, 0) - major axis on y axis = 2a - minor axis on x axis = 2b - vertices at (0, $\pm$ a)	

Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ no relation between a and b	- centre (0, 0) - vertices on x axis - transverse axis between vertices = 2a - conjugate axis = 2b - vertices ($\pm$ a, 0) - asymptotes y = $\pm \frac{b}{a} x$	
Hyperbola	$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$ no relation between a and b	- centre (0, 0) - vertices on y axis - transverse axis between vertices = 2a - conjugate axis = 2b - vertices (0, $\pm$ a) - asymptotes y = $\pm \frac{a}{b} x$	
Equilateral Hyperbola or Rectangular Hyperbola	$x^2 - y^2 = a^2$	- centre (0, 0) - transverse axis on x axis - vertices ($\pm$ a, 0) - asymptotes y = $\pm$ x	
Equilateral Hyperbola or Rectangular Hyperbola	$x^2 - y^2 = -a^2$	- centre (0, 0) - transverse axis on y axis - vertices (0, $\pm$ a) - asymptotes y = $\pm$ x	

Examples:

1. Find the equation of the following conics.

   a) circle centre (0, 0) with radius $\sqrt{5}$.

   $x^2 + y^2 = r^2$ Choose the appropriate equation.

   $x^2 + y^2 = (\sqrt{5})^2$ Substitute in the value of the radius.

   $x^2 + y^2 = 5$ Simplify.

b) parabola symmetric about the x axis through (2, 4) and vertex (0, 0).

$x = ay^2$	Choose the correct parabola equation.
$2 = a(4)^2$	Substitute in values for x and y.
$a = \frac{1}{8}$	Solve for "a".
$x = \frac{1}{8}y^2$	Put "a" back in the original parabola equation.

c) ellipse centre (0, 0) with one vertex at (- 5, 0) and through (- 4, 2).

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Choose the correct ellipse equation.
$\frac{(-4)^2}{5^2} + \frac{2^2}{b^2} = 1$	Substitute in the values for x, y and "a".
$\frac{4}{b^2} = 1 - \frac{16}{25}$	Rearrange to solve for "b".
$\frac{4}{b^2} = \frac{9}{25}$	
$b^2 = \frac{100}{9}$	
$\frac{x^2}{25} + \frac{y^2}{\frac{100}{9}} = 1$ or	Substitute in the values for "a" and "b".
$\frac{x^2}{25} + \frac{9y^2}{100} = 1$	

d) Hyperbola centre (0, 0) , one vertex (0, - 6) and one asymptote y = 3x.

$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$	Choose the correct hyperbola equation.
	a = 6 from the vertex and $\frac{a}{b} = 3$ from y = 3x.
	$\therefore \frac{6}{b} = \frac{3}{1}$ $\therefore$ b = 2
$\frac{x^2}{4} - \frac{y^2}{36} = -1$	Substitute in the values for "a" and "b".

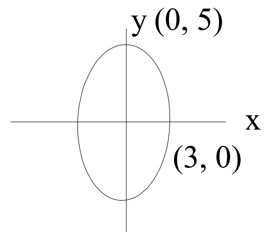
e) Rectangular hyperbola centre (0, 0) with vertex on the y axis through (-2, 3).

$x^2 - y^2 = -a^2$	Choose the correct rectangular hyperbola equation.
$(-2)^2 - 3^2 = -a^2$	Substitute in the values for x and y.
$a^2 = 5$	Solve for a^2.
$x^2 - y^2 = -5$	Substitute back in the value for a^2.

2. Identify each of the following by name, describe each and sketch.

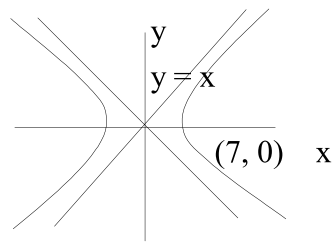
a) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

Ellipse with vertices on
y axis at (0, $\pm$ 5),
major axis = 10 units
minor axis = 6 units
centre (0, 0)



b) $x^2 - y^2 = 49$

Rectangular hyperbola
vertices on the x axis
at ($\pm$ 7, 0)
asymptotes y = $\pm$ x
centre (0, 0)

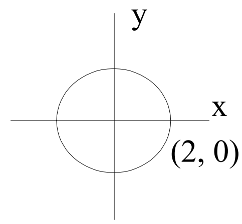


c) $16x^2 + 16y^2 = 64$

Divide both sides by 16.

$$x^2 + y^2 = 4$$

Circle with radius = 2
centre (0, 0)

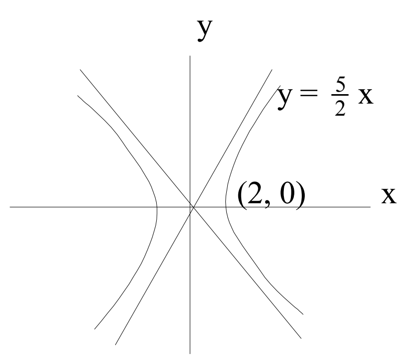


d) $25x^2 - 4y^2 = 100$

Divide both sides by 100

$$\frac{x^2}{4} - \frac{y^2}{25} = 1$$

Hyperbola vertices
on x axis at ($\pm$ 2, 0)
transverse axis = 4
conjugate axis = 10
asymptotes y = $\pm\frac{5}{2}$ x
centre (0, 0)



Translation of Conics - Conics with centre moved to (h, k)

Conics that have been translated have their centre moved from (0, 0) to another point on the x, y plane. They retain their original shape and all the properties of those at (0, 0). Everything is translated exactly the same amount as the centre including the vertices and asymptotes. The following are the equations in standard form for each of the conics with centre (h, k).

Circle

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad (x-h)^2 + (y-k)^2 = r^2$$

Parabola

$$y = ax^2 \quad \Rightarrow \quad (y-k) = a(x-h)^2$$

or

$$x = ay^2 \quad \Rightarrow \quad (x-h) = a(y-k)^2$$

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \Rightarrow \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)}{a^2} = 1$$

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

or

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1 \quad \Rightarrow \quad \frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = -1$$

or

$$x^2 - y^2 = \pm a^2 \quad \Rightarrow \quad (x-h)^2 - (y-k)^2 = \pm a^2$$

Examples:

1. Name and describe each of the following.

   a) $\frac{(x-3)^2}{16} - \frac{(y+2)^2}{49} = -1$

   Hyperbola centre (3, -2) , vertices on x = 3 parallel to y axis at (3, 5) and (3, -9), transverse axis = 14, conjugate axis = 8, asymptotes (y + 2) = $\pm\frac{7}{4}$ (x - 3).

b) $(x+5)^2+(y-1)^2=50$

Circle with centre (- 5, 1) and radius = $\sqrt{50}=5\sqrt{2}$

c) $y=2(x+3)^2-1 \quad \Rightarrow \quad (y+1)=2(x+3)^2$

Parabola opening up with vertex (- 3, - 1)

Equations of Conics in Standard Form

When the equations of conics with centre (h, k) have been expanded the centre and the properties of the conic can not easily be seen. It is necessary to put the conic back into standard form using the method of **completing the square**. The method retrieves the perfect square brackets that show the centre. See Formula for Success page 13 and Unit 2 in the section on Equations the won't factor for more information on completing the square.

Examples:

1. Find the centre of each of the following.

a) $x^2+y^2-10x+4y+20=0$

$x^2-10x+y^2+4y=-20$ Group the x terms together and the y terms also.

$(x^2-10x+25-25)+(y^2+4y+4-4)=-20$

Add and subtract the square of half the coefficient of the x and y terms.

$(x-5)^2-25+(y+2)^2-4=-20$ Factor the first 3 terms in each bracket and remove last term in each bracket.

$(x-5)^2+(y+2)^2=9$ Collect like terms.

$\therefore$ circle with centre (5, -2) and radius 3.

b) $4x^2-9y^2-8x+36y-68=0$

$4x^2-8x-9y^2+36y=68$

$4(x^2-2x+1-1)-9(y^2-4y+4-4)=68$

Factor out the coefficients of x^2 and y^2 then add and subtract the square of half the x and y coefficients

$4(x-1)^2-4-9(y-2)^2+36=68$. Factor the first 3 terms in the brackets and multiply out the last term in each bracket. (Watch the signs)

$4(x-1)^2-9(y-2)^2=36$ Collect like terms.

$\dfrac{(x-1)^2}{9}-\dfrac{(y-2)^2}{4}=1$ Divide by 36.

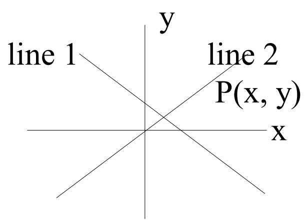
$\therefore$ hyperbola centre (1, 2) with vertices on y = 2 parallel to the x axis.

Solving Systems of Equations

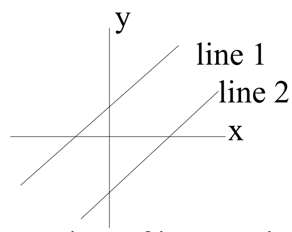
The solution to a system of two equations is the ordered pair(s) (x, y) that are the point or points of intersection. These points will satisfy both the equations. These may be the points where cost and revenue are equal or supply equals demand. The method of solving depends upon the types of equations in the particular question.

Linear Systems - intersection of two lines

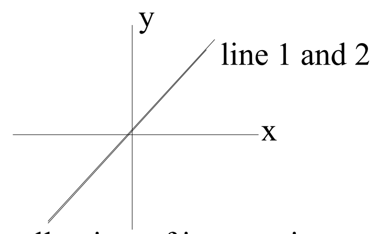
Two lines may intersect in only one point or may not intersect at all if the lines are parallel. If the two equations represent the same line then the points of intersection are all points on the line.



one point of intersection



no points of intersection
lines are parallel



all points of intersection
lines are same

To solve these equations two methods are generally used - elimination or substitution. Either method can be used on most questions but one method may be shorter or easier on particular equation.

<u>Elimination</u> - use addition or subtraction to eliminate one the variables

Examples:

1. $7x + 2y = 23$	**(1)**	The coefficients of one of the variables must be the same
$5x - 4y = 49$	**(2)**	to use elimination.
$14x + 4y = 46$	**(3) = (1)** $\times 2$	Multiply **(1)** by 2 so the y coefficients are the same.
$19x = 95$		Add **(2)** and **(3)** to eliminate y.
$x = 5$		Solve for x.
substitute $x = 5$ into **(1)**		Use the x value in one of the original equations to find y.
$7(5) + 2y = 23$		
$2y = 23 - 35$		
$2y = -12$		
$y = -6$		

$\therefore$ the point of intersection is (5, - 6).

2. $2x - 3y = 8$	**(1)**	
$12x - 18y = 11$	**(2)**	
$12x - 18y = 48$	**(3) = (1)** $\times 6$	Multiply **(1)** by 6. Both x and y coefficients are the same.
$0 = 37$		Subtract **(3) - (2)**

Not true $0 \neq 37$. $\therefore$ no solution.

These lines have the same slope and are parallel. There is no point of intersection.

<u>Substitution</u> - substitute the expression for one variable into the other equation.

Examples:

1. $2x - y = 13$ **(1)**

$3x + 4y = -8$ **(2)**

$y = 2x - 13$ **(1)** Rearrange **(1)** to solve for y.

$3x + 4(2x - 13) = -8$ Substitute this expression into **(2)** for y.

$3x + 8x - 52 = -8$ Expand the bracket and solve for x.

$11x = 44$

$x = 4$

Substitute $x = 4$ into **(1)**

$y = 2(4) - 13$

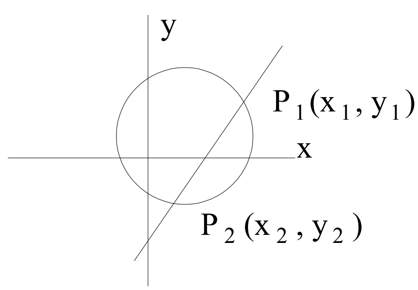
$y = -5$

$\therefore$ the point of intersection is $(4, -5)$.

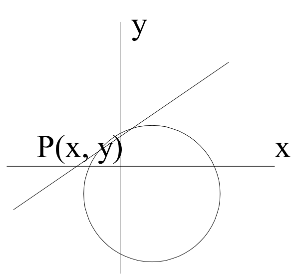
Check your solutions by substituting the solution into the other original equation. If the solution is correct it will satisfy both equations.

Linear - Quadratic Systems - intersection of a line and a conic

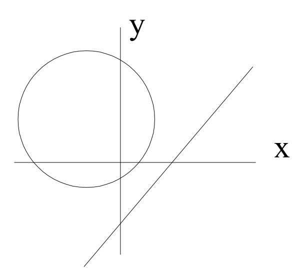
These systems are solved using the method of substitution and may have 0, 1, or 2 solutions.



Two points of contact.
Line is a secant to conic.



One point of contact.
Line is tangent to conic.



No points of contact.

Examples:

1. $x^2 + y^2 = 5$ **(1)**

$y = 2x$ **(2)** A line and a circle.

$\therefore x^2 + (2x)^2 = 5$ Substitute expression for y into **(1)**.

$x^2 + 4x^2 = 5$ Simplify and solve for x.

$5x^2 = 5$

$x^2 = 1$

$x = \pm 1$ Substitute values for x into **(2)**

$y = 2(1)$ or $y = 2(-1)$

$y = 2$ or $y = -2$

$\therefore$ points of intersection are $(1, 2)$ and $(-1, -2)$.

2. $x^2 + 16y^2 = 16$ **(1)**

x - 4y = 4 **(2)**	
x = 4 + 4y **(2)**	Rearrange **(2)** to solve for x.
$(4+4y)^2 + 16y^2 = 16$	Substitute the expression for x into **(1)**.
$16 + 32y + 16y^2 + 16y^2 = 16$	Expand the bracket.
$32y^2 + 32y = 0$	Collect like terms.
32y(y + 1) = 0	Factor. (Common factor)
$\therefore$ y = 0 or y = - 1	Solve for y.
x = 4 + 4(0) or x = 4 + 4(- 1)	Solve for x using **(2)**.
x = 4 or x = 0	

$\therefore$ the points of intersection are (4, 0) and (0, - 1).

Equations of Tangents to Curves - **given the slope of the tangent or a point on the tangent**

Lines that are tangent to one of the conics have only one point of intersection with the curve. That means that there is only one solution to the quadratic equation or that $b^2 - 4ac = 0$ when using the quadratic formula to solve. Since a tangent is a line you must know or be given either the slope of the line or a point on the line.

Examples:

1. Determine the equation(s) of the tangents to $16x^2 + y^2 = 16$ with slope 1.

y = 1x + b **(1)**	General equation to line with slope 1.
$16x^2 + y^2 = 16$ **(2)**	Equation of ellipse.
$16x^2 + (x+b)^2 = 16$	Substitute the expression for y into **(2)**.
$16x^2 + x^2 + 2bx + b^2 = 16$	Square bracket.
$17x^2 + 2bx + b^2 - 16 = 0$	Collect like terms. Quadratic equation with a = 17, b = 2b, and c = $b^2 - 16$.
$b^2 - 4ac = 0$	One solution for tangent.
$(2b)^2 - 4(17)(b^2 - 16) = 0$	Substitute in values for a, b, and c.
$4b^2 - 68b^2 + 1088 = 0$	Simplify.
$64b^2 = 1088$	Collect like terms.
$b^2 = 17$	Solve for "b".
$b = \pm\sqrt{17}$	

$\therefore$ the equations with slope 1 tangent to $16x^2 + y^2 = 16$ are y = x $\pm\sqrt{17}$.

2. Find the equation of the tangents to $x^2 + y^2 = 1$ with y intercept of - 2.

y = mx - 2	**(1)**	General equation to line with y_i = - 2.
$x^2 + y^2 = 1$	**(2)**	Equation of circle.
$x^2 + (mx-2)^2 = 1$		Substitute expression for y into **(2)**.
$x^2 + m^2x^2 - 4mx + 4 = 1$		Square bracket.
$(1+m^2)x^2 - 4mx + 3 = 0$		Collect like terms. Quadratic with a = $1+m^2$, b = - 4m and c = 3.
$b^2 - 4ac = 0$		Tangent has one solution.
$(-4m)^2 - 4(1+m^2)(3) = 0$		Substitute in values for a, b, and c.
$16m^2 - 12 - 12m^2 = 0$		Expand the brackets.
$4m^2 = 12$		Collect like terms.
$m^2 = 3$		Solve for "m".
$m = \pm\sqrt{3}$		

$\therefore$ The equations of the tangents to $x^2 + y^2 = 1$ with y intercept of - 2 are y = $\pm\sqrt{3}$ x - 2.

Limits

The limiting value of a function is the value for f(x) as x approaches a given value.
Example: If f(x) = 2x - 1 find the limiting value of f(x) as x approaches 2. By looking at the graph of f(x) around the x value of 2 we can see that the values of y or f(x) get closer and closer to 3.

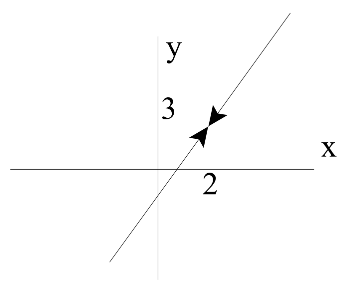
ie. $\lim_{x \to 2} (2x-1)$

= 2(2) - 1

= 3

As x approaches 2

f(x) approaches 3



Limits of Indeterminate Form

When finding the limit, if the graph is continuous (has no gaps or undefined areas) the limit can be found by substituting the value for x into the function as above. If the answer for f(x) is indeterminate (ie. $\frac{0}{0}$) when the value for x is substituted into the question then another approach must be taken. This may take the form of factoring, using a substitute variable or rationalizing the numerator. Whatever manipulation is used, the object is to remove the zero in the denominator when the value of x is substituted into f(x).

Examples:

1. $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$ When - 2 is substituted into the function the answer is $\frac{0}{0}$.

$= \lim_{x \to -2} \frac{(x+2)(x-2)}{(x+2)}$ Factor the numerator and divide out the common factor.

$= \lim_{x \to -2} (x-2)$ Now - 2 can be substituted into the expression.

$= -4$

2. $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$ When 0 is substituted into the function the answer is $\frac{0}{0}$.

Let u = $\sqrt{x+4}$ Use substitution to remove the root.

$\therefore u^2 = x + 4 \quad \Rightarrow \quad x = u^2 - 4$

When x = 0 for the limit u = $\sqrt{4}$ or 2

$\therefore \lim_{u \to 2} \frac{u-2}{u^2 - 4}$ Substitute for x and x^2.

$= \lim_{u \to 2} \frac{u-2}{(u-2)(u+2)}$ Factor and divide out.

$= \lim_{u \to 2} \frac{1}{u+2}$

$= \frac{1}{4}$ Substitute in 2 for u.

3. $\lim_{x \to 0} \frac{\sqrt{1+4x} - \sqrt{1+x}}{x}$ $\frac{0}{0}$ is the answer when 0 is substituted in for x.

$= \lim_{x \to 0} \left(\frac{\sqrt{1+4x} - \sqrt{1+x}}{x} \right) \left(\frac{\sqrt{1+4x} + \sqrt{1+x}}{\sqrt{1+4x} + \sqrt{1+x}} \right)$ Multiply numerator and denominator by the conjugate of the numerator. Same roots but the sign between them is opposite.

$= \lim_{x \to 0} \frac{1 + 4x - (1+x)}{x\left(\sqrt{1+4x} + \sqrt{1+x}\right)}$ Multiply out the numerator and leave the denominator factored.

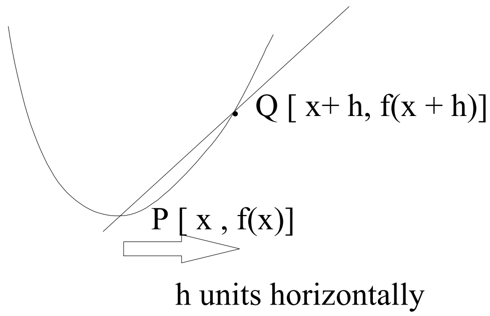
$= \lim_{x \to 0} \frac{3x}{x\left(\sqrt{1+4x} + \sqrt{1+x}\right)}$ Simplify the numerator and then divide out the x.

$= \lim_{x \to 0} \frac{3}{\sqrt{1+4x} + \sqrt{1+x}}$ Substitute in 0 for x.

$= \frac{3}{2}$

The Slope of the Tangent to a Curve - Given a point on the curve

The slope of a tangent line to a curve at a point P is the limiting slope of the secant line PQ as Q slides along the curve towards P. (Note: the points P and Q are on the curve f(x).)



The horizontal distance between P and Q is h units. As Q slides along the curve towards P the distance h approaches zero.

$\therefore$ slope of the secant PQ $= \dfrac{f(x+h)-f(x)}{(x+h)-x}$

$= \dfrac{f(x+h)-f(x)}{h}$

$\therefore$ slope of the tangent at P $= \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

Examples:

1. Find the slope of the tangent to f(x) = x^2 at x = 3.

$\therefore$ the slope of the tangent at P = $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ when f(x) = x^2,

f(x + h) = $(x + h)^2$,
x = 3, and f(3) = 9

$= \lim\limits_{h\to 0}\dfrac{(3+h)^2-9}{h}$

$= \lim\limits_{h\to 0}\dfrac{(3+h-3)(3+h+3)}{h}$ Factor numerator (difference of squares)

$= \lim\limits_{h\to 0}\dfrac{h(6+h)}{h}$ Simplify numerator and divide out h.

= 6 Substitute in 0 for h.

$\therefore$ the slope of the tangent at x = 3 to f(x) = x^2 is 6.

2. Find the slope of the tangent to f(x) = $\sqrt{x}$ when x = 4.

m of tangent $= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ — when f(x) = $\sqrt{x}$, f(x+h) = $\sqrt{x+h}$, x = 4 and f(4) = 2

$= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$ — Let u = $\sqrt{4+h}$, $\therefore$ h = u^2-4

$= \lim_{u\to 2}\frac{u-2}{u^2-4}$ — When h = 0, u = $\sqrt{4}$ or 2.

$= \lim_{u\to 2}\frac{u-2}{(u-2)(u+2)}$ — Factor and divide out the common factor.

$= \lim_{u\to 2}\frac{1}{u+2}$ — Substitute in 2 for u.

$= \frac{1}{4}$

$\therefore$ The slope of the tangent to f(x) = $\sqrt{x}$ at x = 4 is $\frac{1}{4}$.

b) Find the equation of the tangent to f(x) = $\sqrt{x}$ when x = 4.

m = $\frac{1}{4}$ and (4, 2) is on the tangent

$\frac{y-y_1}{x-x_1}=m$ — Use slope form of the equation of a line.

$\frac{y-2}{x-4}=\frac{1}{4}$ — Substitute in values for (x, y) and m.

4(y - 2) = x - 4 — Multiply both sides by the denominators.

4y - 8 = x - 4

x - 4y + 4 = 0 — Rearrange in standard form.

$\therefore$ The equation of the tangent is x - 4y + 4 = 0.

3. Find the slope of the tangent to f(x) = x^3 at the point P [x, f(x)].

m of tangent $= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ — when f(x) = x^3 and f(x + h) = $(x + h)^3$

$= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$

$= \lim_{h\to 0}\frac{\left[(x+h)-x\right]\left[(x+h)^2+x(x+h)+x^2\right]}{h}$ — Factor as a difference of cubes.

$= \lim_{h\to 0}\frac{h\left[(x+h)^2+x(x+h)+x^2\right]}{h}$ — Simplify and divide out h.

$= \lim_{h \to 0} (x+h)^2 + x(x+h) + x^2$

$= (x+0)^2 + x(x+0) + x^2$ Substitute in h = 0.

$= 3x^2$

$\therefore$ The slope of the tangent at all points on the curve f(x) = x^3 is $3x^2$. If the slope is needed at a particular x value it can be substituted into $3x^2$.

The Derivative

The limit $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is so important in the study of calculus that it is given its own name and notation. It is called the derivative and is denoted by $f'(x)$ or y'. These are read as f prime of x or y prime. The derivative and the slope of the tangent to a curve at the point P [x, f(x)] are one and the same thing. Thus the slope of a tangent "m" to a curve is

$$m = f'(x) \quad \text{where}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now try the following questions for practice. Follow the steps as you work your way through to check your answers.

Practice

1. Given the points A (- 2, 3) and B (4, - 5) find the following:

a) the length of the line segment AB.

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Use the distance formula.

$= \sqrt{(4-(-2))^2 + (-5-3)^2}$ Substitute in the values from points A and B.

$= \sqrt{6^2 + (-8)^2}$ Simplify.

$= \sqrt{100}$

$= 10$ Evaluate.

b) the equation of the line through AB

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Find the slope of the line first. Use information

$m = \frac{-8}{6} = \frac{-4}{3}$ from a) above.

$\frac{y - y_1}{x - x_1} = m$ Use the slope form of equation.

$\frac{y-3}{x-(-2)} = \frac{-4}{3}$	Substitute in values for m and (x_1, y_1).
3(y - 3) = - 4(x + 2)	Multiply by the denominators.
3y - 9 = - 4x - 8	
4x + 3y - 1 = 0	Put equation in standard form.

c) the equation of the line through A parallel to the y axis.

x = a	Equation of all lines parallel to y axis.
x = - 2	Substitute in the x coordinate of A.

2. Find the slope, x and y intercepts for the line x - 3y + 15 = 0.

y = mx + b	
3y = x + 15	Rearrange in slope y intercept form.
y = $\frac{x}{3} + 5$	Choose the information from the equation.
m = $\frac{1}{3}$ and b = 5	M is the slope and b is the y intercept.
Let y = 0 in the original equation.	Find x intercept by letting y = 0.
∴ x + 15 = 0	
x = - 15	Solve for x.

∴ The slope is $\frac{1}{3}$, the y intercept is 5 and the x intercept is - 15.

3. Find the equation of the line through (- 5, 7) that is perpendicular to 2x - 3y + 7 = 0.

$y = \frac{2}{3}x + \frac{7}{3}$	Rearrange equation in slope y intercept form.
m = $\frac{2}{3}$ ∴ $\perp m = \frac{-3}{2}$	Use the negative reciprocal of the slope for perpendicular slope.
∴ $\frac{y-7}{x-(-5)} = \frac{-3}{2}$	Use the point and slope for the new equation.
2(y - 7) = - 3(x + 5)	
2y - 14 = -3x - 15	Multiply through by the denominators.
3x + 2y + 1 = 0	Put in standard form.

4. Determine if the point (- 3, 4) is on, inside or outside the circle given by $x^2 + y^2 = 36$.

L.S. = $(-3)^2 + 4^2$ R.S. = 36	Substitute in the x and y values on the left side.
= 9 + 16	
= 25	Evaluate and if the number is more than, less than
25 < 36	or equal to the right side then the point is outside
∴ (- 3, 4) is inside the circle	inside or on the circle.

5. Name and give key features of the conics given by the following equations.

a) $\frac{(x-2)^2}{25} - \frac{(y+3)^2}{49} = 1$	Look at the conic equations and find the appropriate
hyperbola, centre (2, - 3)	one. Negative sign in between is a key feature.
transverse axis parallel to x axis	The positive 1 indicates parallel to x axis.
transverse axis = 10	
conjugate axis = 14	
b) $x = 2(y-7)^2 + 4$	Look at the equations to find the right match.
$(x-4) = 2(y-7)^2$	Move the 4 to be with the x.
parabola, opening right	
vertex (4, 7)	

6. Find the point(s) of intersection for each of the following systems of equations.

a) x + 6y - 17 = 0 **(1)**	
3x - 7y - 1 = 0 **(2)**	
3x + 18y - 51 = 0 **(3) = (1)** × 3	Multiply the first equation by 3 to match x's.
25y - 50 = 0 **(3) - (2)**	Subtract equation 2 from equation 3.
y = 2	Solve for y.
x + 6(2) - 17 = 0	Substitute in 2 for y to find x.
x = 5	Solve for x.

∴ The point of intersection is (5, 2).

b) $x^2 - y^2 = 3$ **(1)**	
x + 2y = 0 **(2)**	
x = -2y **(3)**	Rearrange equation 2 to substitute into equation 1.
$(-2y)^2 - y^2 = 3$	Square and collect like terms.
$3y^2 = 3$	Divide both sides by 3.
$y = \pm 1$	Take square root of both sides to solve for y.
Substitute in y = 1 and -1	Substitute in both values for y in **(2)** to find x.

∴ x = -2(1) or x = -2(-1)
= - 2 = 2

∴ the points of intersection are (- 2, 1) and (2, - 1).

7. Determine the equation of the tangents with slope 2 to $x^2 - y^2 = 9$.

y = 2x + b **(1)**	Equation of all lines with slope 2.
$x^2 - y^2 = 9$ **(2)**	Substitute **(1)** into **(2)**.
$x^2 - (2x+b)^2 = 9$	
$x^2 - (4x^2 + 4bx + b^2) = 9$	Square the bracket.
$-3x^2 - 4bx - b^2 - 9 = 0$	Collect like terms and rearrange to use quadratic

formula where a = -3 , b = - 4b and c = $-b^2 - 9$

$b^2 - 4ac = 0$	One solution as tangent touches one time.
$(-4b)^2 - 4(-3)(-b^2 - 9) = 0$	Substitute in all values for a, b, and c.
$16b^2 - 12b^2 - 108 = 0$	Square and multiply out the bracket.
$4b^2 = 108$	Simplify.
$b^2 = 27$	
$b = \pm\sqrt{27} \Rightarrow \pm 3\sqrt{3}$	Solve for b.

$\therefore$ the equations of the tangents to $x^2 - y^2 = 9$ with slope 2 are y = 2x $\pm 3\sqrt{3}$.

8. Write in standard form to determine the coordinates of the centre of the conic and identify the conic given by the equation $x^2 + 4y^2 - 6x + 1 = 0$.

$(x^2 - 6x + 9 - 9) + 4y^2 = -1$	Rearrange and then add and subtract the square of half the coefficient of the x term.
$(x-3)^2 - 9 + 4y^2 = -1$	Factor the first three terms in the bracket and bring out the subtracted term.
$(x-3)^2 + 4y^2 = 8$	
$\dfrac{(x-3)^2}{8} + \dfrac{y^2}{2} = 1$	Divide by 8 and look at the form to name the conic.

$\therefore$ The conic is an ellipse with centre (3, 0).

9. Find the value of the following limits.

a) $\lim\limits_{x \to -1} \dfrac{x+1}{x^2 - 4x - 5}$	Substitute in -1 and the result is $\frac{0}{0}$.
$= \lim\limits_{x \to -1} \dfrac{x+1}{(x+1)(x-5)}$	Factor the denominator.
$= \lim\limits_{x \to -1} \dfrac{1}{(x-5)}$	Divide out the common factor.
$= \frac{-1}{6}$	Substitute in -1 for x
b) $\lim\limits_{x \to 9} \dfrac{x-9}{\sqrt{x} - 3}$	Substitute in 0 and the result is $\frac{0}{0}$.
$= \lim\limits_{x \to 9} \dfrac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{\sqrt{x} - 3}$	Factor numerator as a difference of squares.
$= \lim\limits_{x \to 9} (\sqrt{x} + 3)$	Divide out the common factor.
$= 6$	Substitute in 9 for x.

9. Find the equation of the tangent to the curve $y = 4 - x^2$ at the point (- 2, 0).

m of the tangent = $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ where f(x) = $4 - x^2$, x = - 2 and f(-2) = 0

$= \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$ f(- 2 + h) = $4 - (-2+h)^2$ = 4 - (4 - 4h + h^2)

$= \lim_{h \to 0} \frac{4h - h^2 - 0}{h}$ $= 4h - h^2$

$= \lim_{h \to 0} \frac{h(4-h)}{h}$ Factor and divide out "h".

= 4 Substitute in 0 for h.

∴ the slope of the tangent to $y = 4 - x^2$ at x = - 2 is 4.

Equation of line with slope 4 through (- 2, 0) is

$\frac{y - 0}{x - (-2)} = 4$ Slope form of equation.

y = 4(x + 2)

y = 4x + 8

∴ The equation of the tangent to the curve $y = 4 - x^2$ at (- 2, 0) is y = 4x + 8.

Review of Unit # 7

1. Given P (3, - 4) and Q (5, 7) find the following:
   a) the slope of PQ.
   b) the length of $|PQ|$
   c) the equation of the line through PQ.
   d) the equation of the line through Q perpendicular to the y axis.

2. Find the x, y intercepts and slope for 3x - 2y + 6 = 0.

3. Find the equation of the line parallel to 3x + 5y - 6 = 0 through (4, - 2).

4. Name and give key features for the following conics:
   a) $x^2 - y^2 = -36$
   b) $y = (x+1)^2 - 4$
   c) $16(x+5)^2 + 9y^2 = 144$
5. Find the equations for the following conic sections:
   a) hyperbola with centre (0, 0), vertices (0, ± 4), the equation of one asymptote y = 2x.
   b) ellipse with centre (- 5, 3), major axis parallel to x axis and length = 24, minor axis length = 18.

6. Find the points of intersection for the equations x + 2y = 7 and $(x-4)^2 + (y+1)^2 = 10$.
7. Determine the equation(s) of the tangent(s) to the curve $x^2 + y^2 = 25$ with y intercept = 6.
8. Change to standard form to determine the coordinates of the centre and identify the conic with the equation $5x^2 - 4y^2 + 20x + 8y - 4 = 0$.
9. Find the following limits:

   a) $\lim\limits_{x \to -1} \dfrac{2x^2 + 5x + 3}{x+1}$

   b) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x}$

10. a) Find the derivative of the curve $y = x^4$.

    b) Determine the slope of the tangent to the curve $y = x^4$ at x = 2.

Answers:

1. a) $\frac{11}{2}$ b) $5\sqrt{3}$ c)11x - 2y - 41 = 0 d) y = 7
2. xi = - 2 yi = 3 m = $\frac{3}{2}$
3. 3x + 5y - 2 = 0
4. a) rectangular hyperbola, centre (0, 0) vertices (0, ± 6), asymptotes y = ± x
   b) Parabola opening up, vertex (- 1, - 4)
   c) Ellipse centre (- 5, 0) major axis parallel to y axis, length = 8, minor axis = 6 vertices (- 5,4) and (- 5, - 4).
5. a) $\dfrac{x^2}{4} - \dfrac{y^2}{16} = -1$ b) $\dfrac{(x+5)^2}{144} + \dfrac{(y-3)^2}{81} = 1$
6. (7, 0) and (3, 2)
7. $y = \pm \frac{\sqrt{11}}{5} x + 6$
8. Hyperbola with centre (- 2, 1) transverse axis = 4, conjugate axis = $2\sqrt{5}$
9. a) 1 b) $\frac{1}{2}$
10. a) $4x^3$ b) m = 32