Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Quiz #8
Etch A Sketch?

Due just before midnight on Thursday, 1 June.*

Please show all your work when answering the question below.

1. Let $f(x) = \dfrac{x}{1-x^2}$. Find the domain, all the intercepts, vertical and horizonal asymptotes, intervals of increase and decrease, maxima and minima, intervals of concavity, and inflection points of $f(x)$. Sketch the graph of $f(x)$ based on this information. *[10]*

SOLUTION. *1. Domain.* $h(x) = \dfrac{x}{1-x^2}$ is defined for all x except where $1 - x^2 = 0$, *i.e.* when $x = \pm 1$. The domain of $h(x)$ is therefore $\{\, x \in \mathbb{R} \mid x \neq \pm 1 \,\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

2. Intercepts. Since $h(0) = \dfrac{0}{1-0^2} = 0$, $h(x)$ has a y-intercept of $y = 0$.

As $h(x) = \dfrac{x}{1-x^2} = 0 \iff x = 0$, $h(x)$ has an x-intercept of $x = 0$. Note that this is also the y-intercept.

3a. Vertical asymptotes. $h(x)$ is continuous and differentiable wherever it is defined, since it is a composition of continuous and differentiable functions, so the only places there might be vertical asymptotes would be at $x = \pm 1$, where $h(x)$ is undefined. We take limits from each side at both of these points to check for vertical asymptotes:

$$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} \frac{x}{1-x^2} \begin{matrix} \to -1^- \\ \to 0^- \end{matrix} = +\infty$$

$$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} \frac{x}{1-x^2} \begin{matrix} \to -1^+ \\ \to 0^+ \end{matrix} = -\infty$$

$$\lim_{x \to +1^-} h(x) = \lim_{x \to +1^-} \frac{x}{1-x^2} \begin{matrix} \to +1^- \\ \to 0^+ \end{matrix} = +\infty$$

$$\lim_{x \to +1^+} h(x) = \lim_{x \to +1^+} \frac{x}{1-x^2} \begin{matrix} \to +1^- \\ \to 0^- \end{matrix} = -\infty$$

It follows that $h(x)$ has vertical asymptotes at both $x = -1$ and $x = +1$. At both points $h(x)$ approaches $+\infty$ from the left and approaches $-\infty$ from the right.

3b. Horizontal asymptotes. We take limits as $x \to \pm\infty$ to check for horizontal asymptotes, with a little help from l'Hôpital's Rule:

$$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{x}{1-x^2} \begin{matrix} \to -\infty \\ \to -\infty \end{matrix} = \lim_{x \to -\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx}\left(1-x^2\right)} = \lim_{x \to -\infty} \frac{1}{-2x} \begin{matrix} \to 1 \\ \to +\infty \end{matrix} = 0^+$$

$$\lim_{x \to +\infty} h(x) = \lim_{x \to +\infty} \frac{x}{1-x^2} \begin{matrix} \to +\infty \\ \to -\infty \end{matrix} = \lim_{x \to +\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx}\left(1-x^2\right)} = \lim_{x \to +\infty} \frac{1}{-2x} \begin{matrix} \to 1 \\ \to -\infty \end{matrix} = 0^-$$

---

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If this fails, you may submit your work to the instructor on paper or by email to `sbilaniuk@ trentu.ca` .

Thus $h(x)$ has $y = 0$ as a horizontal asymptote in both directions, which it approaches from above on the left and from below on the right.

4. Intervals of increase and decrease and maximum and minimum points. As usual, we take the derivative and see what it does:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\frac{x}{1-x^2}\right) = \frac{\left[\frac{d}{dx}x\right)\left(1-x^2\right) - x\left[\frac{d}{dx}\left(1-x^2\right)\right]}{\left(1-x^2\right)^2} = \frac{1\left(1-x^2\right) - x(-2x)}{\left(1-x^2\right)^2} \\ &= \frac{1-x^2+2x^2}{\left(1-x^2\right)^2} = \frac{1+x^2}{\left(1-x^2\right)^2} \end{aligned}$$

$h'(x)$ fails to be defined exactly where $h(x)$ fails to be defined, namely at $x = \pm 1$. Note that since both the numerator and denominator of $h'(x) = \dfrac{1+x^2}{\left(1-x^2\right)^2}$ are positive for all x where $h(x)$ is defined, $h'(x) > 0$ for all $x \neq \pm 1$, and so $h(x)$ is increasing for all $x \neq \pm 1$. Thus $h(x)$ has no critical points and hence no maxima or minima. As usual, we summarize this information in a table:

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$h'(x)$	$+$	undef	$+$	undef	$+$
$h(x)$	$\uparrow$	undef	$\uparrow$	undef	$\uparrow$

5. Intervals of concavity and inflection points. As usual, we compute the second derivative and take it from there:

$$\begin{aligned} h''(x) &= \frac{d}{dx}\left(\frac{1+x^2}{\left(1-x^2\right)^2}\right) = \frac{\left[\frac{d}{dx}\left(1+x^2\right)\right]\left(1-x^2\right)^2 - \left(1+x^2\right)\left[\frac{d}{dx}\left(1-x^2\right)^2\right]}{\left(\left(1-x^2\right)^2\right)^2} \\ &= \frac{2x\left(1-x^2\right)^2 - \left(1+x^2\right)\cdot 2\left(1-x^2\right)\cdot\frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^4} \\ &= \frac{2x\left(1-x^2\right)^2 - 2\left(1+x^2\right)\left(1-x^2\right)(-2x)}{\left(1-x^2\right)^4} = \frac{2x\left(1-x^2\right) + 4x\left(1+x^2\right)}{\left(1-x^2\right)^3} \\ &= \frac{2x - 2x^3 + 4x + 4x^3}{\left(1-x^2\right)^3} = \frac{6x + 2x^3}{\left(1-x^2\right)^3} = \frac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3} \end{aligned}$$

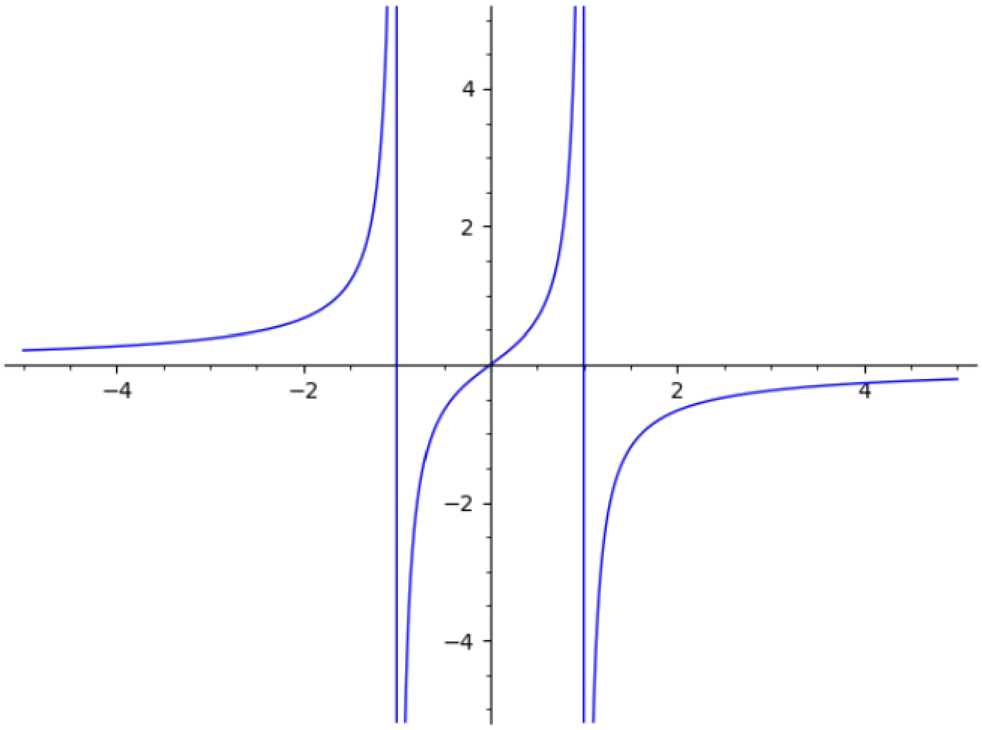
Observe that $h''(x)$ is undefined exactly where $h(x)$ and $h'(x)$ are undefined, namely at $x = \pm 1$. As $3 + x^2 > 0$ for all x, $h''(x) = 0$ exactly when $x = 0$. Since $2x$ is positive or negative exactly as x is positive or negative, and $\left(1-x^2\right)^3$ is positive or negative exactly when $1 - x^2$ is positive or negative, *i.e.* when $-1 < x < 1$ and when $|x| > 1$, respectively, we have that $h''(x) = \dfrac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3}$ is positive when $x < -1$, negative when $-1 < x < 0$, positive when $0 < x < 1$, and negative when $x > 1$. This means that the original function $h(x)$ is concave up when $x < -1$, concave down when $-1 < x < 0$, has an inflection point at $x = 0$, is concave up when $0 < x < 1$, and is concave down when $x > 1$. As usual, we summarize this information in a table:

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$h''(x)$	$+$	undef	$-$	0	$+$	undef	$-$
$h(x)$	$\smile$	undef	$\frown$	infl. pt.	$\smile$	undef	$\frown$

6. Graph. It's a cheat, but here is the graph of $h(x) = \dfrac{x}{1 - x^2}$, as drawn by some program called SageMath:

```
In [1]: plot(x/(1-x^2),-5,5,ymin=-5,ymax=5)
```

Out[1]:



The only interesting point is the origin, which is both the only intercept and the only inflection point, there being no maxima or minima. $\square$