Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2021 (S62)

Solutions to Assignment #5
Oddly Shaped
Due on Friday, 23 July.

Suppose we construct a two-dimesional shape as follows.

At stage 0 we have an equilateral triangle, consisting of three straight line segments of length 1 stuck together end-to-end.

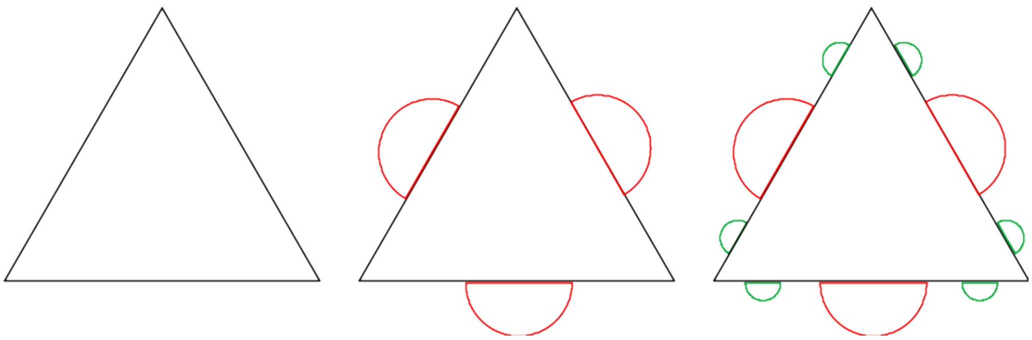
At stage 1 we attach a semi-circle on the outside of the triangle to the middle third of each side, with the base of each semi-circle being that middle third.

At stage 2 we attach a semi-circle on the outside of the triangle to the middle third of each straight piece of a side that is not already the base of a semi-circle, with the base of each new semi-circle being that middle third.

$\vdots$

In general, at stage $n > 0$ we attach a semi-circle on the outside of the triangle to the middle third of each straight piece of a side that is not already the base of a semi-circle, with the base of each new semi-circle being that middle third.

The shape we are interested in is the one we have after completing infinitely many stages, one for each $n \geq 0$. A crude sketch of stages 0 through 2 is given below.



Please answer both questions below, giving your reasoning in detail.

1. What is the length of the perimeter of the shape we have after infinitely many steps, not including the bases of any semi-circles? *[5]*

Solution. *i. Using series.* We'll first count how many line segments we have at the end of each stage and how long they are:

At end of stage	# line segments removed	and # remaining	each of length
0	0	3	1
1	3	6	$\frac{1}{3}$
2	6	12	$\frac{1}{9}$
3	12	24	$\frac{1}{27}$
$\vdots$	$\vdots$	$\vdots$	

It's not too hard to see the pattern: at each stage the number of line segments removed is equal to the number remaining at the end of the previous stage, the number remaining doubles, while the individual length is $\frac{1}{3}$ of what it was at the previous step. In general, at the end of stage $n > 0$, we will have removed $3 \cdot 2^{n-1}$ line segments, each of length $\left(\frac{1}{3}\right)^n = \frac{1}{3^n}$, leaving $3 \cdot 2^n$ line segments of this length.

Now let's count the semicircles and the lengths of their perimeters, not counting the bases. Note that the perimeter or circumference of a circle of radius r is $2\pi r$, so the perimeter, not counting the base, of a semi-circle of radius r is half that, namely πr. Also note that the radius of each semicircle is half the length of the line segment it is replacing.

At end of stage	# of semi-circles added	each of radius	and perimeter
0	0	–	–
1	3	$\frac{1}{6}$	$\frac{\pi}{6}$
2	6	$\frac{1}{18}$	$\frac{\pi}{18}$
3	12	$\frac{1}{54}$	$\frac{\pi}{54}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Observe that the number of semicircles added in stage n is the same as the number of line segments at the end of the previous stage, and each has a radius half the length of the line segments removed/remaining at stage n. In general, by the end of stage $n > 0$, we will have added $3 \cdot 2^{n-1}$ semi-circles, each of radius $\frac{1}{2 \cdot 3^n}$ and hence with perimeter (not counting the base) of $\frac{\pi}{2 \cdot 3^n}$.

Putting the various bits of information together, we start with a perimeter of 3. From this we delete line segments with a total length of

$$3 \cdot \frac{1}{3} + 6 \cdot \frac{1}{9} + 12 \cdot \frac{1}{27} + \cdots = \sum_{n=1}^{\infty} 3 \cdot 2^{n-1} \cdot \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} .$$

This is a geometric series with first term $a = \left(\frac{2}{3}\right)^{1-1} = 1$ and common ratio $r = \frac{2}{3}$, so it converges (since $|r| = \frac{2}{3} < 1$) to $\frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = \frac{1}{1/3} = 3$. Thus we delete line segments with a total length of 3 during the process. We add back the perimeters of the semi-circles added in the process, which amount to

$$3 \cdot \frac{\pi}{6} + 6 \cdot \frac{\pi}{18} + 12 \cdot \frac{1}{54} + \cdots = \sum_{n=1}^{\infty} 3 \cdot 2^{n-1} \cdot \frac{\pi}{2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{\pi}{2} \cdot \frac{2^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{\pi}{2} \left(\frac{2}{3}\right)^{n-1} .$$

This is a geometric series with first term $a = \frac{\pi}{2}\left(\frac{2}{3}\right)^{1-1} = \frac{\pi}{2}$ and common ratio $r = \frac{2}{3}$, so it converges (since $|r| = \frac{2}{3} < 1$) to $\frac{a}{1-r} = \frac{\frac{\pi}{2}}{1-\frac{2}{3}} = \frac{\pi/2}{1/3} = \frac{3\pi}{2}$. Thus we add back a total length of $\frac{3\pi}{2}$ in the form of the perimeters of the semi-circles (not counting theor bases).

It follows that the perimeter of the final shape is $3 - 3 + \frac{3\pi}{2} = \frac{3\pi}{2}$. Whew! □

ii. Without using series. Every piece of straight line is eventually replaced by a collection of semi-circles. Whenever a line segment of length s is replaced, it is replaced by a semi-circle of radius $\frac{s}{2}$ and hence perimeter $\frac{s\pi}{2}$ (not counting the base). Thus the total length of the perimeter is increased by a factor of $\frac{\pi}{2}$ by the end of the process. Since we started with a perimeter of 3, the final shape has perimeter $\frac{3\pi}{2}$. ■

2. What is the total area of the shape we have after infinitely many steps? *[5]*

SOLUTION. The area of an equilateral triangle of side length 1 is $\frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. (You get to work out why, or just look it up ... :-) To this we will be adding the areas of all the semi-circles added in the process. Note that the area of a circle of radius r is πr^2, so the area of a semi-circle of radius r is $\frac{\pi}{2}r^2$.

From the analysis in the series solution to question **1** above, at each stage $n > 0$ we add $3 \cdot 2^{n-1}$ semi-circles, each of radius $\frac{1}{2 \cdot 3^n}$ and hence area $\frac{\pi}{2} \cdot \left(\frac{1}{2 \cdot 3^n}\right)^2 = \frac{\pi}{2} \cdot \frac{1}{2^2 3^{2n}} = \frac{\pi}{8} \cdot \frac{1}{9^n}$. It follows that the total area of the all the semi-circles added in the process is

$$\sum_{n=1}^{\infty} 3 \cdot 2^{n-1} \cdot \frac{\pi}{8} \cdot \frac{1}{9^n} = \sum_{n=1}^{\infty} \frac{3\pi}{8} \cdot \frac{1}{9} \left(\frac{2}{9}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{\pi}{24} \left(\frac{2}{9}\right)^{n-1}.$$

This is a geometric series with first term $a = \frac{\pi}{24}$ and common ratio $r = \frac{2}{9}$, so it converges (since $|r| = \frac{2}{9} < 1$) to $\frac{a}{1-r} = \frac{\frac{\pi}{24}}{1 - \frac{2}{9}} = \frac{\pi/24}{7/9} = \frac{\pi}{24} \cdot \frac{9}{7} = \frac{3\pi}{56}$.

Thus the total area of the final shape is $\frac{\sqrt{3}}{4} + \frac{3\pi}{56}$. ■

[Total = 10]