Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2021 (S62)

Solutions to Assignment #4
Series Business
Due on Friday, 16 July.

Most of the time, finding the actual sum of a series is pretty hard, which is why we usually settle for determining whether the series converges, *i.e.* has a reasonable sum, or not. In this assignment, we will look at a few of the exceptions. Along with the usual tools of algebra, differentiation, and integration, we will rely on the one class of series that has a really nice summation formula, namely geometric series. The geometric series with first term a and common ratio r is the series $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$. As long as $|r| < 1$, or $a = 0$, this series is guaranteed to converge and adds up to $\frac{a}{1-r}$; if $|r| \geq 1$ and $a \neq 0$, it is guaranteed to diverge.

You may use the facts noted above, as well as those developed in the lectures and the textbook, in answering the questions below. You may also assume that you can safely differentiate and integrate series involving powers of x term-by-term when they converge.

1. What is the sum of the series $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$ when it converges? For which values of x does it converge? *[1]*

Solution. The given series is a geometric series with first term $a = 1$ and common ratio $r = -x$, so it sums to $\frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}$ when it converges. Also, as a gemoetric series, it converges when $|r| = |-x| = |x| < 1$, *i.e.* when $-1 < x < 1$, and diverges otherwise, *i.e.* when $x \leq -1$ or $x \geq 1$. ∎

2. What should the the sum of the series $\sum_{n=1}^{\infty} (-1)^n nx^{n-1} = -1 + 2x - 3x^2 + 4x^3 - \cdots$ be when it converges? Why? *[1.5]*

Solution. Using the assumption that we may safely differentiate series term-by-term whe they converge, here is a calculation that tells us how this series should add up, with the help of the formula we obtained in the solution to question **1** above.

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^n nx^{n-1} &= -1 + 2x - 3x^2 + 4x^3 - \cdots = \frac{d}{dx}1 - \frac{d}{dx}x + \frac{d}{dx}x^2 - \frac{d}{dx}x^3 + \frac{d}{dx}x^4 - \cdots \\
&= \frac{d}{dx}\left(1 - x + x^2 - x^3 + \cdots\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n x^n\right) \\
&= \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(1+x)^{-1} = (-1)(1_x)^{-2}\frac{d}{dx}(1+x) \\
&= 1(1+x)^{-2} \cdot 1 = \frac{-1}{(1+x)^2} \qquad \blacksquare
\end{aligned}$$

3. What should the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ be when it converges? Why? *[1.5]*

SOLUTION. Using the assumption that we may safely integrate series term-by-term whe they converge, here is a calculation that tells us how this series should add up, with the help of the formula we obtained in the solution to question **1** above.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \int 1\,dx - \int x\,dx + \int x^2\,dx - \int x^3\,dx + \cdots \\ &= \int \left(1 - x + x^2 - x^3 + \cdots\right) dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^n\right) dx = \int \frac{1}{1+x}\,dx \\ &= \int \frac{1}{u}\,du = \ln(u) + C \quad \text{[Substituting } u = 1+x\text{, so } du = dx.\text{]} \\ &= \ln(1+x) + C\end{aligned}$$

It remains to sort out the constant of integration, C. Plugging in $x = 0$ into the original series and into the final expression tells us that

$$0 = 0 - \frac{0^2}{2} + \frac{0^3}{3} - \frac{0^4}{4} + \cdots = \ln(1+0) + C = 0 + C\,,$$

so $C = 0$. Thus $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \ln(1+x)$ when the series converges. ■

4. What should the sum of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$ be, assuming it converges? *[0.5]*

SOLUTION. This series is what you get when you plug $x = \frac{1}{2}$ into the series given in **1**. (Note that this is within the range of values for which that series converges.) The sum should then be given by the formula obtained in question **1**, namely

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}\,. \qquad \blacksquare$$

5. What should the sum of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ be, assuming it converges? *[0.5]*

SOLUTION. This series is what you get when you plug $x = 1$ into the series given in question **3**. Assuming it converges, it should give you the sum given by the formula obtained question **3**, namely

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(1+1) = \ln(2)\,. \qquad \blacksquare$$

6. Starting with the series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$, give an (informal) chain of reasoning that computes the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, assuming it converges. *[5]*

SOLUTION. The series $\sum_{n=0}^{\infty} x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots = \sum_{n=0}^{\infty} \left(-x^2\right)^n$ is a geometric series with first term $a = 1$ and common ratio $r = -x^2$. It follows that it converges when $|r| = \left|-x^2\right| = x^2 < 1$, *i.e.* when $-1 < x < 1$, and diverges otherwise, *i.e.* when $x \leq -1$ or $x \geq 1$. When it converges the sum is given by the formula $\frac{a}{1-r} = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$.

Similarly to the calculation in the solution to **3** above, we will integrate both the given series and its sum formula, although we'll lay it out in a different order than for **3**:

$$\begin{aligned}
\arctan(x) + C &= \int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right) dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx \\
&= \int 1\,dx - \int x^2\,dx + \int x^4\,dx - \int x^6\,dx + \int x^8\,dx - \cdots \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}
\end{aligned}$$

As in the solution to **3**, we need to determine the constant of integration C. If we plug in $x = 0$ at both ends, we get

$$C = 0 + C = \arctan(0) + C = 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \frac{0^9}{9} - \cdots = 0\,,$$

so we should have $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \arctan(x)$ when the series converges.

The numerical series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, which converges by the Alternatibf Series Test, is what we get we plug $x = 1$ into $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$. It should therefore be (and is!) the case that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan(1) = \frac{\pi}{4}$. ■

NOTE: The power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ summing to $\arctan(x)$ obtained above is usually called *Gregory's series* after the Scottish mathematician and astronomer James Gregory (1638-1675) who rediscovered it in 1668. It had previously been discovered by the Indian mathematician and astronomer Madhava of Sangamagrama (1350-1410), and was independently rediscovered in 1676 by Gottfried Leibniz (1646-1716), one of the co-inventors of calculus.

[Total = 10]