Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2021 (S62)

Assignment #3
Kosh says its a cinch.
Due on Friday, 9 July.

The functions $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$, mentioned in the lecture *Integration by Parts II*, are the basic *hyperbolic* functions, analogously to the basic trigonometric functions, $\cos(x)$ and $\sin(x)$. (The other hyperbolic functions are defined in terms of the basic ones in the same way that the other trigonometric functions are defined in terms of the basic one.) Their names are pronounced "kosh" and "sinch", respectively.

Since the basic hyperbolic functions are each others' derivatives (with no gratuitous negative signs) and satisfy a reasonably nice identity, namely $\cosh^2(x) - \sinh^2(x) = 1$ (sadly with a negative), they are sometimes used in place of the trigonometric functions when making substitutions. They have other uses in mathematics as well: they are needed to help do trigonometry in certain curved spaces, they arise in solving various differential equations (see question **3** below), and they turn out to be intimately related to the standard trigonometric functions. (For example, $\cos(x) = \cosh(ix)$ and $\cosh(x) = \cos(ix)$, where $i = \sqrt{-1}$.)

1. Verify that $\cosh^2(x) - \sinh^2(x) = 1$ for all x. *[1]*

SOLUTION. Here goes:

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{\left(e^x\right)^2 + 2e^xe^{-x} + \left(e^{-x}\right)^2}{4} - \frac{\left(e^x\right)^2 - 2e^xe^{-x} + \left(e^{-x}\right)^2}{4} \\
&= \frac{\left(e^x\right)^2 + 2e^xe^{-x} + \left(e^{-x}\right)^2 - \left(e^x\right)^2 + 2e^xe^{-x} - \left(e^{-x}\right)^2}{4} \\
&= \frac{4e^xe^{-x}}{4} = \frac{4e^{x-x}}{4} = \frac{4e^0}{4} = \frac{4 \cdot 1}{4} = 1 \quad \blacksquare
\end{aligned}$$

2. Work out what the inverse function of $\sinh(x)$, let's call it $\operatorname{arcsinh}(x)$, is in terms of more common functions. *[3]*

SOLUTION. Since $y = \operatorname{arcsinh}(x) \iff x = \sinh(y) = \dfrac{e^y - e^{-y}}{2}$, we need to solve the latter equation for y. Here we go:

$$\begin{aligned}
x = \sinh(y) &\iff x = \frac{e^y - e^{-y}}{2} \iff 2x = e^y - e^{-y} = e^y - \frac{1}{e^y} \\
&\iff 2xe^y = \left(e^y\right)^2 - 1 \iff \left(e^y\right)^2 - 2xe^y - 1
\end{aligned}$$

At the last step above we have reached a quadratic equation in the expression e^y, so we apply the quadratic formula.

$$\begin{aligned} e^y &= \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{2x \pm \sqrt{4x^2+4}}{2} \\ &= \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1} \end{aligned}$$

Since $e^y > 0$ for all values of y and $x - \sqrt{x^2+1} < 0$ for all values of x, we can discard the "solution" $e^y = x - \sqrt{x^2+1}$ to the original quadratic equation. It follows that $e^y = x + \sqrt{x^2+1}$, and thus

$$\operatorname{arcsinh}(x) = y = \ln\left(x + \sqrt{x^2+1}\right) .$$

Note that since $x + \sqrt{x^2+1} > 0$ for all x, as $|x| < \sqrt{x^2+1}$ for all x, we have that $\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ is defined for all x. ■

3. Suppose $y = f(x)$ is a functions satisfying the the differential equation $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, and also satisfies the initial conditions $f(1) = f(-1) = \cosh(1)$. Show that it must be the case that $f(x) = \cosh(x)$. *[6]*

NOTE THE FIRST: It is easy to check that $\cosh(x)$ satisfies the differential equation and the initial conditions. Why is it the only function that does?

Hint: Let $z = \frac{dy}{dx}$, so $\frac{dz}{dx} = \frac{d^2y}{dx^2}$. Rewrite the equation in terms of z, move everything involving x to one side of the new equation and everything involving z to the other), and integrate to solve for z. Then get y by ...

NOTE THE SECOND: This differential equation would arise if you suspended a certain length of chain from the points specified by the initial conditions and let it hang under the influence of gravity, "down" being the negative y direction, and asked what shape the chain would have if no other forces were in play.

SOLUTION. Following the hint, let $z = \frac{dy}{dx}$, so $\frac{dz}{dx} = \frac{d^2y}{dx^2}$. The given equation $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ then becomes $\frac{dz}{dx} = \sqrt{1+z^2}$. We will try to solve this equation for z as a function of x, and then integrate $z = \frac{dy}{dx}$ to solve for y. The initial conditions will be used after that to pin down the constants of integration as much as we can.

The basic techniques we will use to solve the differential equation $\frac{dz}{dx} = \sqrt{1+z^2}$ is *separation of variables*: treating $\frac{dz}{dx}$ as a fraction, put everything involving x on one side and everything involving z on the other and integrate each side with respect to the variable appearing on that side. This is, incidentally, one of the several places in calculus where you get away with treating an expression like $\dfrac{dz}{dx}$ as if it really were a fraction, which is generally a no-no. The expression is officially shorthand for a certain *limit*, after all. It really upsets pure mathematicians like myself that this works! Engineers, physicists, applied mathematicians, and such, just dont care because it works ... :-) Anyway, this first part is pretty easy:

$$\frac{dz}{dx} = \sqrt{1+z^2} \implies \frac{dz}{\sqrt{1+z^2}} = dx \implies \int \frac{1}{\sqrt{1+z^2}}\, dz = \int 1\, dx$$

One side of the integral equation is pretty trivial to integrate; for the other side, we use the trigonometric substitution $z = \tan(\theta)$, so $dz = \sec^2(\theta)\, d\theta$ and $\sec(\theta) = \sqrt{1+z^2}$. Here we go:

$$\begin{aligned} x = \int 1\, dx &= \int \frac{1}{\sqrt{1+z^2}}\, dz = \int \frac{1}{\sqrt{1+\tan^2(\theta)}} \sec^2(\theta)\, d\theta = \int \frac{1}{\sqrt{\sec^2(\theta)}} \sec^2(\theta)\, d\theta \\ &= \int \sec(\theta)\, d\theta = \ln\left(\tan(\theta) + \sec(\theta)\right) + C = \ln\left(z + \sqrt{1+z^2}\right) + C \end{aligned}$$

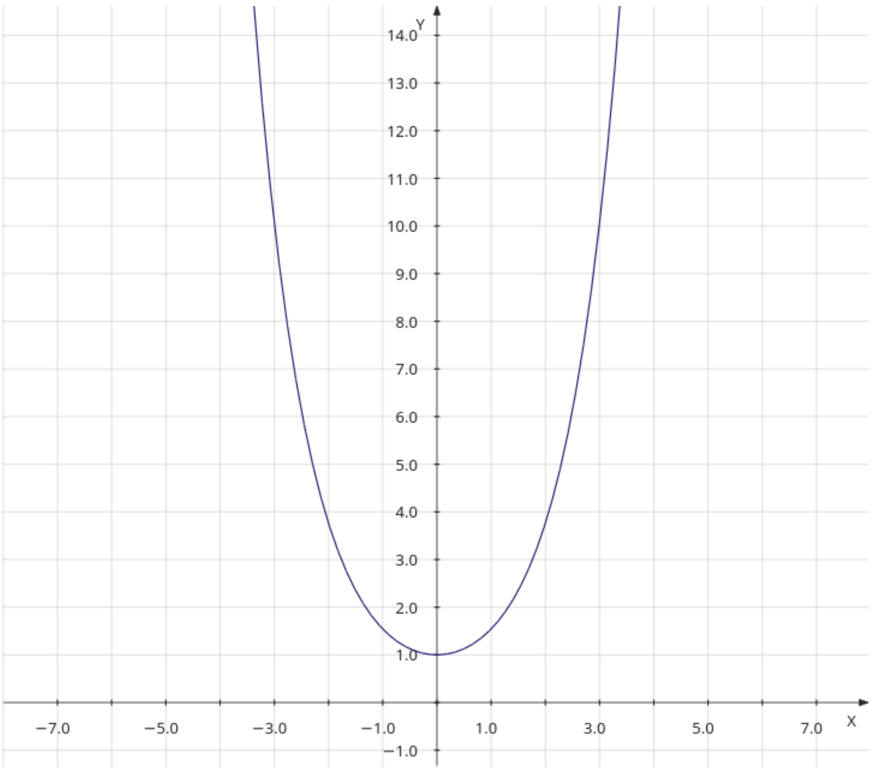
We now have to solve $x = \ln\left(z + \sqrt{1+z^2}\right) + C$ for z in terms of x:

$$\begin{aligned} x = \ln\left(z + \sqrt{1+z^2}\right) + C &\implies x - C = \ln\left(z + \sqrt{1+z^2}\right) \\ &\implies e^{x-C} = e^{\ln\left(z+\sqrt{1+z^2}\right)} = z + \sqrt{1+z^2} \\ &\implies e^{x-C} - z = \sqrt{1+z^2} \implies \left(e^{x-C} - z\right)^2 = \left(\sqrt{1+z^2}\right)^2 \\ &\implies \left(e^{x-C}\right)^2 - 2e^{x-C}z + z^2 = 1 + z^2 \\ &\implies \left(e^{x-C}\right)^2 - 2e^{x-C}z = 1 \implies -2e^{x-C}z = -\left(e^{x-C}\right)^2 + 1 \\ &\implies z = \frac{-\left(e^{x-C}\right)^2 + 1}{-2e^{x-C}} = \frac{\left(e^{x-C}\right)^2 - 1}{2e^{x-C}} \\ &\implies z = \frac{\left(e^{x-C}\right)^2 - 1}{2e^{x-C}} \cdot \frac{1/e^{x-C}}{1/e^{x-C}} = \frac{e^{x-C} - \frac{1}{e^{x-C}}}{2} \\ &\implies z = \frac{e^{x-C} - e^{-(x-C)}}{2} = \sinh(x - C) \end{aligned}$$

Thus $\frac{dy}{dx} = z = \sinh(x-C)$, where C is an unknown constant. We integrate this to recover the solution y of the given differential equation as a function of x, using the substitution $u = x - C$, so $du = dx$. Since another constant of integration will be appearing and we are already using C, the new constant of integration will be called K. Recall that cosh and sinh are each other's derivatives, and hence also each other's antiderivatives.

$$y = \int \frac{dy}{dx}\, dx = \int \sinh(x - C)\, dx = \int \sinh(u)\, du = \cosh(u) + K = \cosh(x - C) + K$$

It remains to find the values of the constants C and K. This is where we – finally! – use the initial conditions that when $x = \pm 1$, we have $y = \cosh(1)$. we will also exploit the facts that $\cosh(-t) = \dfrac{e^{-t} + e^{-(-t)}}{2} = \dfrac{e^{t} + e^{-t}}{2} = \cosh(t)$ for all t, and that $\cosh(t)$ is decreasing for $t < 0$ and increasing for $t > 0$, as is obvious from its graph:



Of course, one could also get the latter fact by analyzing the behaviour of the derivative.

First, observe that when $x = \pm 1$, we have $\cosh(1 - C) + K = \cosh(1) = \cosh(-1) = \cosh(-1-C)+K$. Subtracting K from both ends get us that $\cosh(1-C) = \cosh(-1-C)$. Since cosh is an even function that is decreasing before 0 and increasing after 0, $\cosh(a) = \cosh(b)$ is only possible when $a = \pm b$, so it follows that either $1 - C = -1 - C$ or $1 - C = -(-1 - C) = 1 + C$. The equation $1 - C = -1 - C$ makes no sense because if you add C to both sides you would get that $1 = -1$. The latter equation we can solve for C: $1 - C = 1 + C \implies 2C = 0 \implies C = 0$. Thus $y = \cosh(x - 0) + K = \cosh(x) + K$.

Second, since $y = \cosh(x) + K$ and $y = \cosh(1)$ when $x = 1$, we have that $\cosh(1) = \cosh(1) + K$, from which it follows immediately that $K = 0$.

Thus if $y = f(x)$ is the solution to the given differential equation with the given initial conditions, we must have that $y = f(x) = \cosh(x)$. Whew! ■

[Total = 10]