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Taylor Series III - Taylor polynomials and remainder terms

(§11.11 in the textbook.)

Recall: The Taylor series at a of $f(x)$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. (Almost always equal to $f(x)$ when it converges.)

Def'n: The Taylor polynomial of $f(x)$ at a of degree $k \geq 0$ is the polynomial

$$\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

[Note that any particular Taylor polynomial is defined for all x, since it's a polynomial.]

The difference between $f(x)$ and $T_k(x)$ is the remainder $R_k(x) = f(x) - T_k(x)$.

There are various ways to write $R_k(x)$ without direct reference to the difference.

eg <u>Theorem:</u> (Lagrange form of the remainder).
Suppose that $f(x)$ and all of its derivatives up to and including $f^{(k+1)}(x)$ are defined in some interval $(a-r, a+r)$ centred at a.
Then for each x with $a-r < x < a+r$ (and $x \neq a$) there is a z between a and x such that

$$R_k(z) = f(x) - T_k(x) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}.$$

(There are other ways to write a formula for $R_k(z)$, but this one is often the easiest to use.)

There are examples in the text for using this to figure out how to approximate functions using Taylor polynomials to achieve a given accuracy.

We'll use this to compute the value of $e = e^1$ (3)
to within $\frac{1}{100} = 0.01$.

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $\Leftarrow$ The Taylor series of e^x at $a=0$.

$$T_k(x) = \sum_{n=0}^{k} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^k}{k!}$$

e^x and its Taylor series (and $T_k(x)$) are defined and (infinitely differentiable) for all x, ie $-\infty < x < +\infty$.

By the theorem above, it follows that for any $x \neq 0$ we have a z with $0 < z < x$ (or $x < z < 0$) such that

$$R_k(x) = e^x - T_k(x) = \frac{e^z}{(k+1)!} \cdot x^{k+1},$$

With $x=1$, this means that there is a $0 < z < 1$
s.t. $R_k(1) = e - T_k(1) = \frac{e^z}{(k+1)!} 1^{k+1} = \frac{e^z}{(k+1)!}$

but since e^x is an increasing function: $\leq \frac{e^1}{(k+1)!} < \frac{3}{(k+1)!}$

So $R_k(1) < \frac{3}{(k+1)!}$, so if we make

$$\frac{3}{(k+1)!} < \frac{1}{100} = 0.01$$

we guarantee that $T_k(1)$ is within 0.01 of $e^1 = e$.

$$\frac{3}{(k+1)!} < \frac{1}{100} \iff \underbrace{3/\tfrac{1}{100}}_{300} < (k+1)!$$

so we need a k s.t. $(k+1)! > 300$

k	$(k+1)!$
0	$1! = 1$
1	$2! = 2$
2	$3! = 6$
3	$4! = 24$
4	$5! = 120$
5	$6! = 720 > 300$

so $k=5$ will suffice

Thus $T_5(1)$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\Big|_{x=1}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$$

is within 0.01 of e.

$$= 1 + 1 + 0.5 + 0.1\dot{6} + 0.041\dot{6} + 0.008\dot{3}$$

$$\approx 1 + 1 + 0.5 + 0.167 + 0.042 + 0.008$$

$= 2.717$

so $e \approx 2.717$ according to this and

e is actually equal to $2.718\ldots$, which is

certainly within 0.01 of the approximation.

Next time: We'll use this technology to show that e is an irrational number (ie it can't be written precisely as a ratio of integers).