Power Series (§ 11.8 in the textbook)

A power series is an expression of the form $\sum_{n=0}^{\infty} a_n x^n$, where x is a variable & each a_n is a constant (as far as x is concerned).

[In some parts of mathematics, power series are manipulated as algebraic objects without regard for convergence. eg in combinatorics (generating functions) & in parts of statistics]

We'll mainly be concerned with them as functions of x, so for us convergence matters. Usually, we'll want to know for which values of x the power series converges.

eg The prototypes for power series are geometric series:

$$a+ax+ax^2+\cdots = \sum_{n=0}^{\infty} ax^n$$, where a is a constant.

We know from looking at geometric series, that this will converge (for all x) if $a=0$ or if $|x|<1$, in which case it converges to $\frac{a}{1-x}$, and diverges otherwise.

eg Exponential series: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
$(0! = 1)$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

(this turns out to be equal to e^x)

When does this converge? Use the Ratio Test first:

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| x \cdot \frac{1}{n+1} \right|$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1,$$

so the series converges (absolutely) for all x.

eg $\Rightarrow$ $\displaystyle\sum_{n=0}^{\infty} \frac{(n+3)^2}{2^n} x^n$ Do the Ratio Test first:

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1+3)^2}{2^{n+1}}\cdot x^{n+1}}{\frac{(n+3)^2 x^n}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+4)^2}{(n+3)^2}\cdot\frac{\cancel{2^n}}{2^{n+1}}\cdot\frac{x^{\cancel{n+1}}}{\cancel{x^n}}\right|$$

$$= |x| \lim_{n\to\infty} \frac{1}{2}\cdot\left(\underbrace{\frac{n+4}{n+3}}_{1}\right)^2 = \frac{|x|}{2}$$

since $\displaystyle\frac{n+4}{n+3}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1+\overbrace{\frac{4}{n}}^{\to 0}}{1+\underbrace{\frac{3}{n}}_{\to 0}} \Rightarrow 1$ as $n\to\infty$

This tells us, by the Ratio Test, that the series converges when $\frac{|x|}{2} < 1$, ie for $-2 < x < 2$, and diverges when $\frac{|x|}{2} > 1$, ie for $x < -2$ or $x > 2$. When $\frac{|x|}{2} = 1$, ie when $x = \pm 2$, the Ratio Test tells us nothing, so we need check these points separately:

$x = 2$: $\displaystyle\sum_{n=0}^{\infty} \frac{(n+3)^2}{\cancel{2^n}}\cdot\cancel{2^n} = \sum_{n=0}^{\infty} (n+3)^2$ which diverges by the Divergence Test since $\displaystyle\lim_{n\to\infty}(n+3)^2 = \infty$ ...

$x = -2$: $\displaystyle\sum_{n=0}^{\infty} \frac{(n+3)^2}{2^n}(-2)^n = \sum_{n=0}^{\infty} (-1)^n (n+3)^2$ which also diverges by the Divergence Test.

e) $\lim_{n\to\infty} \frac{(-1)^n 2^n}{n!} x^n$ Use the Ratio Test first:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 2^{n+1} x^{n+1}}{(n+1)!}}{\frac{(-1)^n 2^n x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-2x}{n+1} \right| = 2|x| \lim_{n\to\infty} \frac{1}{n+1} = 2|x| \cdot 0 = 0 < 1$$

so this one also converges for all x by the Ratio Test.

f) $\sum_{n=0}^{\infty} \frac{2^n x^n}{3^n}$ We'll use the Root Test this time.

$$\lim_{n\to\infty} \left| \frac{2^n x^n}{3^n} \right|^{1/n} = \lim_{n\to\infty} \frac{2^{n \cdot \frac{1}{n}}}{3^{n \cdot \frac{1}{n}}} \cdot |x|^{n \cdot \frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2}{3}|x| = \frac{2}{3}|x|$$

when $\frac{2}{3}|x| < 1$

$\Leftrightarrow |x| < \frac{3}{2}$

So, by the Root Test, the series converges (absolutely) for $-\frac{3}{2} < x < \frac{3}{2}$ & diverges when $x < -\frac{3}{2}$ or $x > \frac{3}{2}$.

When $x = \pm\frac{3}{2}$ we have to use other tests:

$x = -\frac{3}{2}$: $\sum_{n=0}^{\infty} \frac{2^n}{3^n} \cdot \left(-\frac{3}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$

which diverges by the Divergence Test since $\lim_{n\to\infty} (-1)^n$ does not exist.

$x = \frac{3}{2}$: $\sum_{n=0}^{\infty} \frac{2^n}{3^n} \cdot \left(\frac{3}{2}\right)^n = \sum_{n=0}^{\infty} 1$

which diverges by the Divergence Test since $\lim_{n\to\infty} 1 = 1 \neq 0$.

<u>Pattern:</u> A power series $\sum_{n=0}^{\infty} a_n x^n$ will usually (absolutely) converge for all x inside some interval of the form $(-R, R)$ [where $R = \infty$] and diverges for $|x| > R$, and may converge or diverge for $x = \pm R$. This R is the <u>radius of convergence</u> of the powerseries.

eg $\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$ Throw the Ratio Test at this:

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}x^{n+1}}{(n+1)+1}}{\frac{2^n x^n}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{x^{n+1}}{x^n}\cdot\frac{n+1}{n+2}\right|$$

$$= 2|x|\cdot\lim_{n\to\infty}\frac{n+1}{n+2} = 2|x|\cdot\lim_{n\to\infty}\frac{n+1}{n+2}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = 2|x|\lim_{n\to\infty}\frac{1+\overset{\to 0}{\frac{1}{n}}}{1+\underset{\to 0}{\frac{2}{n}}} = 2|x|$$

Thus the series converges absolutely for $2|x|<1$, ie for $-\frac{1}{2}<x<\frac{1}{2}$, and diverges for $2|x|>1$, ie $x<\frac{-1}{2}$ or $x>\frac{1}{2}$, by the ~~Root~~ Ratio Test. This means that the radius of convergence of this series is $R=\frac{1}{2}$. So what happens at ~~R~~ $\pm\frac{1}{2}$?

$x=-\frac{1}{2}$: $\sum_{n=0}^{\infty}\frac{2^n(-\frac{1}{2})^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$,

which is the alternating harmonic series, which converges, conditionally by the Alternating Series Test.

[check for yourself if you don't remember.]

$x = \frac{1}{2}$: $\sum_{n=0}^{\infty} \frac{2^n (\frac{1}{2})^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$,

which is the harmonic series, which diverges by the p-Test since $p = 1 \leq 1$.

So $\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$ converges for x in the interval $[-\frac{1}{2}, \frac{1}{2})$ & diverges outside it.

Note: It is possible that if the radius of convergence of a power series is R, for the series to converge on $(-R, R)$

[if $R < \infty$] or $[-R, R)$

or $(-R, R]$

or $[-R, R]$.

abs. cvgce. on $(-R, R)$ and (at best, usually) conditional cvgce. at the endpts,

Next: express arbitrary functions as power series.