2021-07-07

Average Value of a Function (§9.4) and the Arc-length of a Curve (§9.9)

(We'll skip §9.5, 9.6, & 9.8 in the text and have already done all we're going to of §9.7 - the material on improper integrals.)

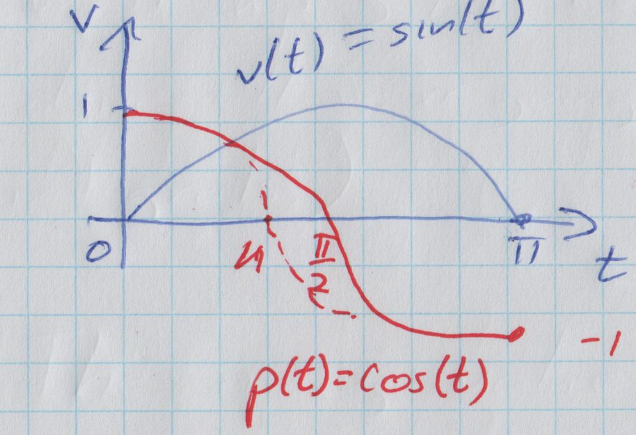
The average value of $f(x)$ on $[a,b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

(Analog of $\frac{a_1+\dots+a_n}{n}$ to get an average.)

eg Imagine that $v(t)$ gives speed of travel in a straight line and you travel from $t=a$ to $t=b$. The average speed is $\frac{p(b)-p(a)}{b-a}$ ← change in position / ← change in time, but $p(t)$ is the anti-derivative of velocity so the Fundamental Thm of Calculus tells us that $p(b)-p(a)=\int_a^b v(t)\,dt$.

eg Suppose we take a short trip in a straight line with speed at time t being given by $v(t) = \sin(t)$, (km/hr) where $0 \le t \le \pi$.



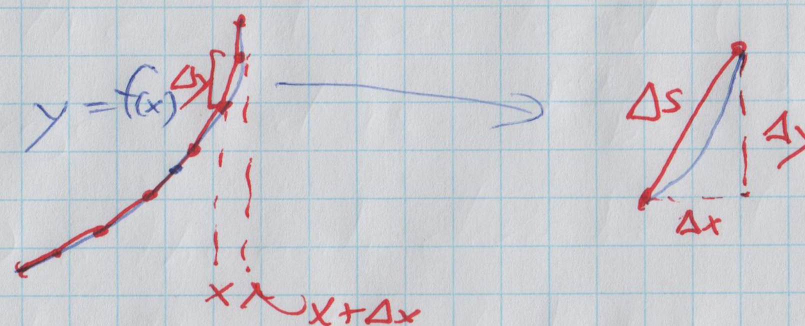
$$\text{Average speed} = \frac{\int_0^{\pi} \sin(t)\,dt}{\pi - 0}$$

$$= \frac{1}{\pi}\left(-\cos(t)\right)\Big|_0^{\pi}$$

$$= \frac{-1}{\pi}\cos(\pi) + \frac{1}{\pi}\cos(0)$$

$$= \frac{-1}{\pi}(-1) + \frac{1}{\pi}\cdot 1 = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi} \text{ km}$$

Arc-length of a curve $y = f(x)$:



Pythagoras tells us that $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$

$$\Rightarrow \Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x$$

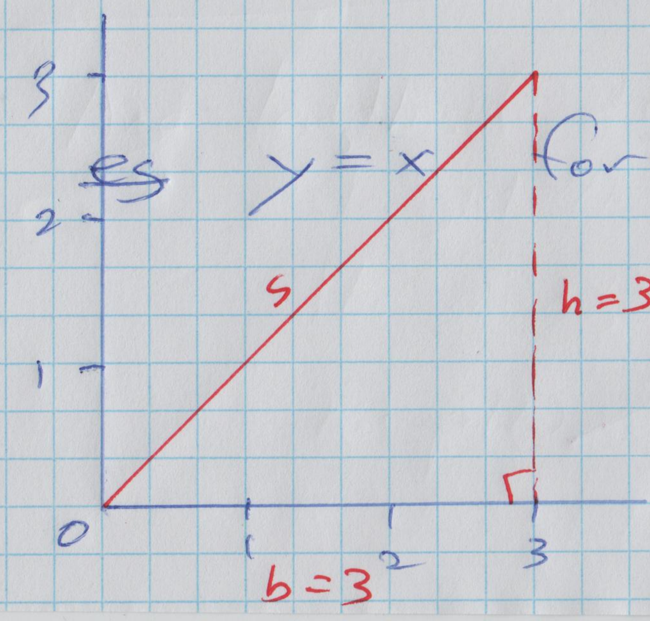
As we shrink Δx to 0, this gives us the infinitesimal bit of arc-length of the curve being given by

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

To get the arc-length of $y = f(x)$ from $x = a$ to $x = b$, you integrate:

$$\text{Arc-length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1+(f'(x))^2}\, dx$$



for $0 \le x \le 3$ $\quad \frac{dy}{dx} = \frac{dx}{dx} = 1$

$$s = \sqrt{b^2 + h^2}$$
$$= \sqrt{3^2 + 3^2}$$
$$= \sqrt{18} = \sqrt{9 \cdot 2}$$
$$= 3\sqrt{2}$$

$$\text{Arc-length} = s$$
$$= \int_0^3 1\, ds = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_0^3 \sqrt{1+1^2}\, dx = \int_0^3 \sqrt{2}\, dx$$
$$= \sqrt{2} \cdot x \Big|_0^3 = \sqrt{2} \cdot 3 - \cancel{\sqrt{2} \cdot 0}$$
$$= 3\sqrt{2}.$$

4

<u>es</u> $y = x^2 \qquad 0 \le x \le 2 \qquad \frac{dy}{dx} = \frac{d}{dx}x^2 = 2x$

$$\text{Arc-length} = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1+(2x)^2}\,dx$$

$$= \int_0^2 \sqrt{1+4x^2}\,dx \qquad x = \frac{1}{2}\tan(\theta), \quad dx = \frac{1}{2}\sec^2(\theta)\,d\theta$$

$$= \int_{x=0}^{x=2} \sqrt{1+\cancel{4}\cdot\frac{1}{\cancel{4}}\tan^2(\theta)}\cdot\sec^2(\theta)\,d\theta \qquad 2x = \tan(\theta)$$

$$\sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+4x^2}$$

(since $1+\tan^2(\theta) = \sec^2(\theta)$) $$= \int_{x=0}^{x=2} \sqrt{\sec^2(\theta)}\cdot\sec^2(\theta)\,d\theta$$

(apply the reduction $$= \int_{x=0}^{x=2} \sec^3(\theta)\,d\theta$$

formula to get) $$= \left[\frac{1}{3-1}\tan(\theta)\sec^{3-2}(\theta) - \frac{3-2}{3-1}\int \sec^{3-2}(\theta)\,d\theta\right]_{x=0}^{x=2}$$

$$= \left[\frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\int \sec(\theta)\,d\theta\right]\Bigg|_{x=0}^{x=2}$$

$$= \left[\frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\ln\left(\sec(\theta) + \tan(\theta)\right)\right]\Bigg|_{x=0}^{x=2}$$

$$= \left[\frac{1}{\cancel{2}}(\cancel{2}x)\sqrt{1+4x^2} - \frac{1}{2}\ln\left(2x + \sqrt{1+4x^2}\right)\right]\Bigg|_{0}^{2}$$

$$= \left[2\cdot\sqrt{1+4\cdot 2^2} - 2\cdot\sqrt{1+4\cdot 0^2}\right]$$

$$- \left[\frac{1}{2}\ln\left(2\cdot 2 + \sqrt{1+4\cdot 2^2}\right) - \frac{1}{2}\ln\left(2\cdot 0 + \sqrt{1+4\cdot 0^2}\right)\right]$$

$$= \left[2\sqrt{17} - 2\right] - \left[\frac{1}{2}\ln\left(4 + \sqrt{17}\right) - \cancel{\frac{1}{2}\ln(0+1)}\right]$$

$$= 2\sqrt{17} - 2 - \frac{1}{2}\ln\left(4 + \sqrt{17}\right)$$

... and that's the length of the piece of the parabola $y = x^2$ for $0 \le x \le 2$.

Note: the complexity of arc-length integrals rises very rapidly with the complexity of the function, with some rare exceptions

Here is one of the exceptions:

$$y = \cosh(x) = \frac{e^{x}+e^{-x}}{2} \qquad \frac{dy}{dx} = \frac{e^{x}-e^{-x}}{2} = \sinh(x)$$

for $0 \le x \le \ln(2)$ Fact: $1+\sinh^2(x) = \cosh^2(x)$

$$\text{Arc-length} = \int_0^{\ln(2)} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\ln(2)} \sqrt{1+\sinh^2(x)}\, dx = \int_0^{\ln(2)} \sqrt{\cosh^2(x)}\, dx$$

$$= \int_0^{\ln(2)} \cosh(x)\, dx = \sinh(x)\Big|_0^{\ln(2)} = \sinh(\ln(2)) - \sinh(0)$$

$$= \frac{e^{\ln(2)} - e^{-\ln(2)}}{2} - \frac{e^{0}-e^{-0}}{2} = \frac{2-\frac{1}{2}}{2} - \cancel{\frac{1-1}{2}} = \frac{3/2}{2}$$

$$= \boxed{\frac{3}{4}}$$