2021-06-24 ①

Trigonometric Substitutions - handling quadratics inside square roots (§8.3 in the text)

Simple example: $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

Replace x by a trig function of a new variable, to take advantage of a trig identity that will get rid of the root.

θ lower case Greek letter theta

$\sin^2(\theta) = 1-\cos^2(\theta)$

So replace x by $\cos(\theta)$ ...

$$x=\cos(\theta) \Rightarrow dx = \left[\frac{d}{d\theta}\cos(\theta)\right]d\theta = -\sin(\theta)\,d\theta$$

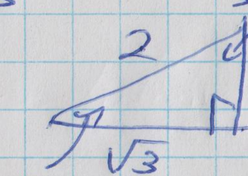
x	θ
0	$+\frac{\pi}{2}$
$\frac{1}{2}$	$\frac{\pi}{3}$

$60^\circ = \frac{\pi}{3}$

$$= \int_{\pi/2}^{\pi/3} \frac{1}{\sqrt{1-\cos^2(\theta)}}\cdot(-\sin(\theta))\,d\theta$$

$$= \cancel{-}\int_{\pi/3}^{\pi/2} \frac{\cancel{-}\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}\,d\theta = \int_{\pi/3}^{\pi/2}\frac{\sin(\theta)}{\sqrt{\sin^2(\theta)}}\,d\theta$$

$$= \int_{\pi/3}^{\pi/2}\frac{\sin(\theta)}{\sin(\theta)}\,d\theta = \int_{\pi/3}^{\pi/2} 1\,d\theta = \theta\Big|_{\pi/3}^{\pi/2} = \frac{\pi}{2}-\frac{\pi}{3} = \boxed{\frac{\pi}{6}}$$



$\frac{\pi}{6} = 30^\circ$

$\cos\left(\frac{\pi}{3}\right) = \frac{adj.}{hyp.} = \frac{1}{2}$

A slightly more example: $\int \sqrt{4+x^2}\, dx$

We'll take advantage of $1+\tan^2(\theta) = \sec^2(\theta)$

First, substitute $x = 2u \Rightarrow u = \frac{x}{2}$ and then $u = \tan(\theta)$,

(Or all at once with $x = 2\tan(\theta)$.)

$dx = 2du$

$$= \int \sqrt{4+(2u)^2}\, 2du = 2\int \sqrt{4+4u^2}\, du = 2\int \sqrt{4(1+u^2)}\, du$$

$u = \tan(\theta)$
$du = \sec^2(\theta)d\theta$
$\sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+u^2}$

$$= 2\int \sqrt{4}\sqrt{1+u^2}\, du = 2\int 2\sqrt{1+u^2}\, du$$

$$= 4\int \sqrt{1+\tan^2(\theta)} \cdot \sec^2(\theta)d\theta = 4\int \sqrt{\sec^2(\theta)} \cdot \sec^2(\theta)d\theta$$

$$= 4\int \sec(\theta)\cdot\sec^2(\theta)d\theta = 4\int \sec^3(\theta)\, d\theta.$$ & now we have a trig integral...

$n=3$
Use the reduction formula...

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$$= 4\left[\frac{1}{3-1} \tan(\theta) \sec^{3-2}(\theta) + \frac{3-2}{3-1} \int \sec^{3-2}(\theta)\, d\theta \right]$$

$$= 4\left[\frac{1}{2} \tan(\theta) \sec(\theta) + \frac{1}{2} \int \sec(\theta)\, d\theta \right]$$

$$= 2 \tan(\theta) \sec(\theta) + 2 \ln(\sec(\theta) + \tan(\theta)) + C$$

$$= \cancel{2\sqrt{1+u^2}\,u}\; 2u\sqrt{1+u^2} + 2\ln\left(u + \sqrt{1+u^2}\right) + C$$

$$= 2 \cdot \frac{x}{2} \sqrt{1+\left(\frac{x}{2}\right)^2} + 2 \ln\left(\frac{x}{2} + \sqrt{1+\left(\frac{x}{2}\right)^2}\right) + C$$

$$= x\sqrt{1+\frac{x^2}{4}} + 2 \ln\left(\frac{x}{2} + \sqrt{1+\frac{x^2}{4}}\right) + C$$

An even more complicated one: (4)

$$\int (4x^2-9)^{-1/2}\,dx = \int \frac{1}{\sqrt{4x^2-9}}\,dx$$

Here we aim to use $\sec^2(\theta)-1 = \tan^2(\theta)$

Let $x = \frac{3u}{2}$, so $dx = \frac{3}{2}\,du$.

$$= \int \left(4\cdot\left(\frac{3u}{2}\right)^2 - 9\right)^{-1/2}\cdot\frac{3}{2}\,du$$

$$= \frac{3}{2}\int\left(\cancel{4}\cdot\frac{9u^2}{\cancel{4}} - 9\right)^{-1/2} du = \frac{3}{2}\int\left(9(u^2-1)\right)^{-1/2} du$$

$$= \frac{3}{2}\int 9^{-1/2}(u^2-1)^{-1/2}\,du = \frac{3}{2}\int \frac{1}{\sqrt{9}}(u^2-1)^{-1/2}\,du$$

$3 = \sqrt{9}$

$$= \frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{3}}\int (u^2-1)^{-1/2}\,du = \frac{1}{2}\int\left(\sec^2(\theta)-1\right)^{-1/2}\sec(\theta)\tan(\theta)\,d\theta$$

$u = \sec(\theta)$

$du = \sec(\theta)\tan(\theta)\,d\theta$

$$= \frac{1}{2}\int\left(\tan^2(\theta)\right)^{-1/2}\sec(\theta)\tan(\theta)\,d\theta$$

$$= \frac{1}{2}\int \frac{1}{(\tan^2(\theta))^{1/2}} \sec(\theta)\tan(\theta)\,d\theta$$

$$= \frac{1}{2}\int \frac{1}{\cancel{\tan(\theta)}} \cdot \sec(\theta)\,\cancel{\tan(\theta)}\,d\theta$$

$u = \sec(\theta)$

$\sqrt{u^2-1} = \sqrt{\sec^2(\theta)-1} = \sqrt{\tan^2(\theta)} = \tan(\theta)$

$$= \frac{1}{2}\int \sec(\theta)\,d\theta = \frac{1}{2}\ln\left(\sec(\theta)+\tan(\theta)\right)+C$$

$$= \frac{1}{2}\ln\left(u+\sqrt{u^2-1}\right)+C$$

$x = \frac{3}{2}u \Rightarrow u = \frac{2}{3}x$

$$= \frac{1}{2}\ln\left(\frac{2}{3}x+\sqrt{\left(\frac{2}{3}x\right)^2-1}\right)+C$$

$$= \frac{1}{2}\ln\left(\frac{2}{3}x+\sqrt{\frac{4}{9}x^2-1}\right)+C$$

$$= \frac{1}{2}\ln\left(\frac{2}{3}x+\frac{1}{3}\sqrt{4x^2-9}\right)+C$$

Quick summary:

1° see $\sqrt{x^2-1}$, try $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)\,d\theta$

2° see $\sqrt{x^2+1}$, try $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$

3° see $\sqrt{1-x^2}$, try $x = \sin(\theta)$, so $dx = \cos(\theta)\,d\theta$

If you see $\sqrt{\pm a^2 \mp b^2x^2}$ then first substitute $x = \frac{b}{a}u$ to get to one of the forms above.

Q.: How do we handle more generic quadratics inside the square root, eg $\int \sqrt{x^2+3x+17}\,dx$?