2021-06-22, ①

# Trigonometric Integrals – the reduction formulas

[§8.2 in the text – we'll be doing more ...]

Last time, we used integration by parts to get the "reduction formula"

reduced power of sin

1° $\displaystyle\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$

This works as long as $n \geq 2$.

Similarly,

2° $\displaystyle\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.$ $(n \geq 2)$

3° $\displaystyle\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x)\cancel{\sec(x)} - \cancel{\frac{n-2}{n-1}}\int \tan^{n-2}(x)\,dx$ $(n \geq 2)$

4° $\displaystyle\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$ $(n \geq 2)$

... all of which can be got by using integration by parts.

We'll derive 3° using parts as an example:

Suppose $n \geq 2$ and we are given $\int \tan^n(x)\,dx$.

How do we split it up into parts?

$$\cancel{u = \tan^{n-1}(x) \qquad v' = \tan(x)}$$

$$\cancel{u' = (n-1)\tan^{n-2}(x) \cdot \sec^2(x)}$$

$$\cancel{v = -\ln(\cos(x))}$$

& now what?

We'll use the fact that

$$1 + \tan^2(x) = \sec^2(x)$$

so $\tan^2(x) = \sec^2(x) - 1$.

Thus $\boxed{\int \tan^n(x)\,dx} = \int \tan^{n-2}(x)\tan^2(x)\,dx$

$$= \int \tan^{n-2}(x)\,(\sec^2(x) - 1)\,dx$$

$$= \int \tan^{n-2}(x)\sec^2(x)\,dx - \int \tan^{n-2}(x)\,dx$$

Substitute $u = \tan(x)$, so $du = \sec^2(x)\,dx$.

$$= \int u^{n-2}\,du - \int \tan^{n-2}(x)\,dx$$

$$= \frac{u^{n-1}}{n-1} - \int \tan^{n-2}(x)\,dx = \boxed{\frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx.}$$

We'll now try to get 4° to check if it's right:

$(n \geq 2)$ $\boxed{\int \sec^n(x)\,dx}$

We'll use parts, with $u = \sec^{n-2}(x)$

& $v' = \sec^2(x)$

so $u' = (n-2)\sec^{n-3}(x)\cdot\sec(x)\tan(x)$

$= (n-2)\sec^{n-2}(x)\tan(x)$

& $v = \tan(x)$.

$$= \sec^{n-2}(x)\tan(x) - \int (n-2)\sec^{n-2}(x)\tan(x)\tan(x)\,dx$$

$$= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,dx$$

$\tan^2(x) = \sec^2(x) - 1$

$$= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)(\sec^2(x) - 1)\,dx$$

$$= \sec^{n-2}(x)\tan(x) - (n-2)\boxed{\int \sec^n(x)\,dx} + (n-2)\int \sec^{n-2}(x)\,dx$$

$$\Rightarrow \overset{n-1}{\overset{\shortparallel}{(n-2+1)}}\int \sec^n(x)\,dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx$$

$$\Rightarrow \boxed{\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx}$$

**Moral:** Look these formulas up as necessary or have a better memory than I do...

**Examples:**

0° $\int \cos^3(x)\,dx$ can be done with the reduction formula
(from last time) $(n=3)$

$$= \frac{1}{3}\cos^{3-1}(x)\sin(x) + \frac{3-1}{3}\int \cos^{3-2}(x)\,dx$$

$$= \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\int \cos(x)\,dx$$

$$= \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\sin(x) + C$$

... like last time.

1° $\int \cos^4(x)\,dx$

Here the idea of replacing $\cos^2(x)$ by $1-\sin^2(x)$ doesn't help...

$$= \int \cos^2(x)\left(1-\sin^2(x)\right)dx$$

$$= ??$$

... but the reduction formula works: $(n=4)$

$$= \cancel{\int}\ \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx$$

... and we'll apply the formula again... $(n=2)$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left[\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int \overset{=1}{\boxed{\cos^0(x)}}\,dx\right]$$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left[\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x\right] + C$$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\cos(x)\sin(x) + \frac{3}{8}x + C$$

<u>Note</u>: We could also handle $\int \cos^2(x)\,dx$ by using a rearranged double angle formula $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

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2° $\displaystyle\int_{-\pi/4}^{\pi/4} \tan^3(x)\,dx$

Recall: $\tan\left(\frac{\pi}{4}\right) = 1$

& $\tan\left(\frac{-\pi}{4}\right) = -1$

$\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

& $\sec\left(\frac{-\pi}{4}\right) = \sqrt{2}$

$(n=3$

$$= \frac{1}{2}\tan^2(x)\Big|_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4} \tan(x)\,dx$$

As with parts, the reduction formulas use the same limits as the original definite integral.

$$= \left(\frac{1}{2}\tan^2\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan^2\left(\frac{-\pi}{4}\right)\right) - \left(-\ln(\cos(x))\right)\Big|_{-\pi/4}^{\pi/4}$$

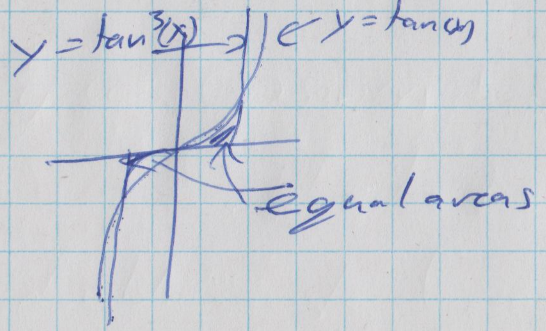
$$= \left(\cancel{\frac{1}{2}\cdot 1^2} - \cancel{\frac{1}{2}\cdot(-1)^2}\right) - \left(\ln\left(\cos^{-1}(x)\right)\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= 0 - \ln(\sec(x))\Big|_{-\pi/4}^{\pi/4}$$

$$= -\ln\left(\sec\left(\frac{\pi}{4}\right)\right) - \left(-\ln\left(\sec\left(\frac{-\pi}{4}\right)\right)\right)$$

$$= -\ln(\sqrt{2}) + \ln(\sqrt{2})$$

$$= 0$$



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3° $\int \sin^4(x)\cos^2(x)\,dx$

$\cos^2(x)+\sin^2(x)=1$

$\Rightarrow \cos^2(x) = 1-\sin^2(x)$

$= \int \sin^4(x)\left(1-\sin^2(x)\right)dx$

$= \int \sin^4(x)\,dx - \int \sin^6(x)\,dx$

& now we could apply the reduction formula for $\int \sin^n(x)\,dx$ ... except that it's very tedius if you start with higher powers, because you have to apply the formula repeatedly.

Next time: What can we do without the reduction formulas?

Sometimes more...