A bit more Substitution, and a start on Integration by Parts.

Note: We added $\int \tan(x)\,dx = \boxed{-\ln(\cos(x)) + C}$

last time to our list of antiderivatives

$$= (-1)\cdot\ln(\cos(x)) + C$$
$$= \ln([\cos(x)]^{-1}) + C$$
$$= \ln\left(\frac{1}{\cos(x)}\right) + C$$
$$= \boxed{\ln(\sec(x)) + C}$$

Now we'll compute $\int \sec(x)\,dx$ [using a very dirty trick]:

Usual try...

$$\int \sec(x)\,dx \overset{\downarrow}{=} \int \frac{1}{\cos(x)}\,dx =$$ Now what? What do we substitute for? $\cos(x)$ is in the denominator and there is no $\sin(x)$ available...

$$\int \sec(x)\,dx = \int \sec(x) \cdot \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$ [What!?]

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

but $\frac{d}{dx}\tan(x) = \sec^2(x)$

& $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$,

so we can substitute $u = \sec(x) + \tan(x)$

so $du = [\sec(x)\tan(x) + \sec^2(x)]\,dx$

$$= \int \frac{1}{u}\,du = \ln(u) + C$$

$$= \ln(\sec(u) + \tan(x)) + C$$

Moral: Sometimes we need to algebra before we can substitute, It may not be obvious what to do...

Memorize this one

or look it up as necessary...

Integration by Parts

... let's us handle products of dissimilar functions in many cases.

It's a converse to (a rearranged form of) the ~~Chain~~ Product ~~Rule~~ Rule for derivatives.

~~Recall: The Chain Rule says that~~

$$\cancel{(f\circ g)'(x) = \frac{d}{dx}\left[f(g(x))\right]}$$

$$\cancel{= f'(g(x))\cdot g'(x)}$$

Recall: The Product Rule says that

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \left[\frac{d}{dx}f(x)\right]\cdot g(x) + f(x)\cdot\left[\frac{d}{dx}g(x)\right]$$

ie $(fg)'(x) = f'(x)g(x) + f(x)\cdot g'(x)$

If we let $u = f(x)$ & $v = g(x)$, this looks like $(uv)' = u'v + u\cdot v'$.

We can rearrange this as

$$u \cdot v' = (u \cdot v)' - u' \cdot v,$$

& integrating on both sides gives us the Integration by Parts formula:

$$\int u \cdot v' \, dx = \int (u \cdot v)' \, dx - \int u' \cdot v \, dx$$
$$= u \cdot v - \int u' \cdot v \, dx$$

Example: $\int x e^x \, dx$. We have to split xe^x into u and v'.

Try $u = x$ $\quad v' = e^x$

so $u' = 1$ $\quad v = e^x$.

$$= \overset{u \cdot v}{x \cdot e^x} - \int \overset{u' \cdot v}{1 \cdot e^x} \, dx = xe^x - \int e^x \, dx$$
$$= xe^x - e^x + C$$
$$= (x-1)e^x + C$$

Convention: the generic constant appears when the last integral sign disappears

<u>Warning:</u> If you're not careful, Integration by Parts, can make things <u>worse</u>:

$$\int x e^x \, dx \qquad \begin{array}{l} \text{Try } u = e^x \ \& \ v' = x, \\ \text{so } u' = e^x \ \& \ v = \frac{x^2}{2}. \end{array}$$

$$= \overset{u \cdot v}{e^x \cdot \frac{x^2}{2}} - \underbrace{\overset{u' \cdot v}{\int e^x \cdot \frac{x^2}{2} \, dx}}_{\text{This is more complicated than what you started with.}}$$

<u>Basic rule of thumb</u> for integration by parts:

Choose u & v' so that

$\int u' \cdot v \, dx$ is simpler (or at least no worse)

than $\int u \cdot v' \, dx$ ...

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1° $\int \ln(x)\,dx$

<u>Problem:</u> this doesn't look like a product!

<u>Solution:</u> make it into one ...

$= \int 1 \cdot \ln(x)\,dx$

Put $u = \ln(x)$ & $v' = 1$,

so $u' = \frac{1}{x}$ & $v = x$.

$\underbrace{\text{simpler} \qquad \text{more complex}}_{\text{overall: not worse?}}$

$= \overset{v \cdot u}{x \cdot \ln(x)} - \int \overset{u' \cdot v}{\frac{1}{x} \cdot x}\,dx$

$= x\ln(x) - \int 1\,dx$

$= \boxed{x\ln(x) - x + C}$

$= x(\ln(x) - 1) + C$

2° Similarly, we can compute

$$\int \arctan(x)\,dx \quad \text{by taking } u = \underset{\tan^{-1}(x)}{\arctan(x)} \ \& \ v' = 1$$

so $u' = \frac{1}{1+x^2}$ & $v = X$

$$= \overset{v \cdot u}{x\arctan(x)} - \int \overset{u' \cdot v}{\frac{1}{1+x^2} \cdot x}\, dx$$

Substitute $w = 1+x^2$, so $dw = 2x\,dx$ & $\frac{1}{2}dw = x\,dx$

$$= x\arctan(x) - \int \frac{1}{w} \cdot \frac{1}{2}\, dw$$

$$= x\arctan(x) - \frac{1}{2}\ln(w) + C$$

$$= \boxed{x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$$

Next time: More integration by parts, including a much more detailed rule for splitting up the integrand.