The Substitution Rule (in its basic form)

... but first, a quick review of where we're at:

Our project is to develop a body of integration techniques and a library of specific antiderivatives to let us compute as many integrals as possible.

Techigues so far:

1° $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ [& similarly for indefinite integrals]

2° $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ [— " —]

3° $\int_a^a f(x)\,dx = 0$

4° $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5° $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(Power Rule) 6° $\int_a^b x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b$ if $n \neq -1$ $\left(= \ln(x)\Big|_a^b \text{ if } n = -1\right)$

7° (Substitution Rule)

$$\int_a^b f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

x	u
a	$g(a)$
b	$g(b)$

$(F'(u)=f(u))$

$$= \int_{x=a}^{x=b} f(u)\,du = F(u)\Big|_{x=a}^{x=b} = F(g(x))\Big|_a^b$$

[& similarly for indefinite integrals, where it's customary to put the antiderivative back in terms of the original variable]

Library (of specific integrals) so far:

0° (Power Rule) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$

$= \ln(x) + C$ if $n = -1$.

1° $\int \cos(x)\,dx = \sin(x) + C$

$\int \sin(x)\,dx = -\cos(x) + C$

We'll be adding $\int \sec(x)\,dx$ & $\int \tan(x)\,dx$ today.

[$\int \csc(x)\,dx$ & $\int \cot(x)\,dx$ are not used that often and are usually looked up. They similarly to the ones for sec(x) & tan(x).]

3° $\int e^x\,dx = e^x + C$ $\qquad \int a^x\,dx = \dfrac{a^x}{\ln(a)} + C$

$(a>0)$

Substitution tells us why:

$u = \ln(a)\cdot x$

$du = \ln(a)\,dx$

$$\int a^x\,dx = \int \left(e^{\ln(a)}\right)^x dx = \int e^{\ln(a)\cdot x}\,dx$$

$$= \int e^u\,du = e^u + C = e^{\ln(a)x} + C$$

$$= \left(e^{\ln(a)}\right)^x + C = a^x + C \checkmark$$

Examples of substitution:

1° $\displaystyle\int_0^1 x^3 e^{x^2}\,dx = \int_0^1 x\cdot x^2\cdot e^{x^2}\,dx$

$\displaystyle = \int_1^e u\, e^u \cdot \frac{1}{2}\,du$

$\displaystyle = \frac{1}{2}\int_1^e u\, e^u\,du$

Simplify by substituting $u = x^2$ so $du = 2x\,dx$ & thus $\frac{1}{2}du = x\,dx$.

x	$u = e^{x^2}$
0	$e^{0^2} = e^0 = 1$
1	$e^{1^2} = e^1 = e$

What now? We'll need another technique of integration to handle $\int u\, e^u\,du$, namely Integration by Parts. [Coming soon!]

Moral: Simplification by Substitution may not be enough, so we'll develop techniques to handle other situations.

2° $\displaystyle\int_0^{\pi/4} \tan(x)\,dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx = \int_0^{\pi/4} \frac{1}{\cos(x)} \sin(x)\,dx$

Substitute $u = \cos(x)$, so $du = -\sin(x)\,dx$

& $(-1)\,du = \sin(x)\,dx$, and also we'll substitute

back & use the old limits.

$$= \int_{x=0}^{x=\pi/4} \frac{1}{u}\cdot(-1)\,du = -\int_{x=0}^{x=\pi/4} \frac{1}{u}\,du = -\int_{x=0}^{x=\pi/4} u^{-1}\,du$$

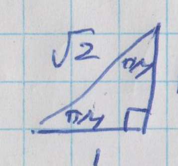
$$= \ln(u)\Big|_{x=0}^{x=\pi/4} = \ln(\cos(x))\Big|_0^{\pi/4}$$

$$= \ln\left(\cos\left(\tfrac{\pi}{4}\right)\right) - \ln(\cos(0))$$

$$= \ln\left(\tfrac{1}{\sqrt{2}}\right) - \ln(1)$$

$$= \ln\left(2^{-1/2}\right) - 0$$

$$= -\tfrac{1}{2}\cdot\ln(2)$$



$\cos(\pi/4) = \frac{1}{\sqrt{2}}$

As usual, we measure angles in terms of radians by default.

Moral (of both examples):

Substitution relies on spotting opportunities. (This takes practice to do quickly & reliably.)

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3° $\int \sqrt{2x+3}\,dx = \int (2x+3)^{1/2}\,dx$ Try $u = 2x+3$, so $du = 2dx$ & thus $\frac{1}{2}du = dx$

$= \int u^{1/2} \cdot \frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2} \cdot \frac{u^{1/2+1}}{\frac{1}{2}+1} + C = \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \cancel{\frac{1}{2}} \cdot \frac{\cancel{2}}{3} \cdot u^{3/2} + C$

$= \boxed{\frac{1}{3}(2x+3)^{3/2} + C} = \frac{1}{3}\left(\sqrt{2x+3}\right)^3 + C$

4° $\int \cos(x)\sqrt{2\sin(x)+3}\,dx$ $u = \sin(x)$ $du = \cos(x)\,dx$

$= \int \sqrt{2u+3}\,du$

& now substitute again: $w = 2u+3$, so $dw = 2du$ & $\frac{1}{2}dw = du$.

$= \int w^{1/2} \cdot \frac{1}{2}\,dw$

$= \frac{1}{2} \cdot \frac{w^{3/2}}{3/2} + C = \frac{1}{3}w^{3/2} + C = \frac{1}{3}(2u+3)^{3/2} + C$

$= \frac{1}{3}(2\sin(x)+3)^{3/2} + C$

We could also have done this in one go:

$$\int \cos(x)\sqrt{2\sin(x)+3}\,dx$$

$w = 2\sin(x)+3$, so

$dw = (2\cos(x)+0)\,dx$

$= 2\cos(x)\,dx$, & thus

$\frac{1}{2}\,dw = \cos(x)\,dx$

$$= \int w^{1/2}\cdot\frac{1}{2}\,dw = \frac{1}{2}\cdot\frac{w^{3/2}}{3/2}+C = \frac{1}{3}w^{3/2}+C$$

$$= \frac{1}{3}(2\sin(x)+3)^{3/2}+C$$

Moral: If you end up doing more than one substitution in a row, you could have done it in one substitution (which is more complicated).

It's probably more reliable to do it in digestible pieces (one little substitution at a time) if you're just starting. As you get better, you combine them more reliably. Practice!