The Ratio and Root Tests (§11.7)

(Especially useful when dealing with power series - §11.8.)

<u>Ratio Test</u>: Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

1) If $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

2) If $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.
   (includes $L = \infty$)

3) If $L = 1$, then the test is inconclusive.
   (The series might converge conditionally, or converge absolutely, or diverge.)

This test tends to work most easily when the terms a_n are built using multiplication and/or division.

<u>Example</u>: $\sum_{n=0}^{\infty} \frac{n}{2^n}$ Trying the Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \left(\frac{n+1}{n}\right) = \frac{1}{2}\lim_{n\to\infty} \left(1 + \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}\right) = \frac{1}{2} \cdot 1 < 1$$

$\therefore$ the series converges absolutely by the Ratio Test.

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n!}$ Try Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}5^{n+1}}{(n+1)!}}{\frac{(-1)^n 5^n}{n!}}\right| = \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n}$$

$$= \lim_{n\to\infty} 5\cdot\underbrace{\frac{1}{n+1}}_{\to 0} = 5\cdot 0 = 0 < 1,$$ so the

series converges absolutely

Example: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ (Alternating Harmonic Series)

– we know this converges by the Alt. Series Test, but $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-Test $(p=1-0=1 \not> 1)$, so the series cvgs. conditionally

Try the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}}\right|$$

$$= \lim_{n\to\infty}\frac{1}{n+1}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}} = \frac{1}{1+0} = 1$$

so the Ratio Test tells us nothing.

Example: For which values of x does the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ converge.

[This is a geometric series with first term $a=1$ and common ratio $r=\frac{x}{2}$, so it converges exactly when $\left|\frac{x}{2}\right|<1$ ie for $-2<x<2$, & diverges ow.]

Try the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{2}\right| = \frac{1}{2}\lim_{n\to\infty}|x| = \frac{|x|}{2}$$

So, by the Ratio Test, the series converges absolutely when $\frac{|x|}{2}<1$ & diverges when $\frac{|x|}{2}>1$. It tells us nothing when $\frac{|x|}{2}=1$ ie when $x=\pm 2$.

Since the series $\sum_{n=0}^{\infty}\frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n$

& $\sum_{n=0}^{\infty}\frac{2^n}{2^n} = \sum_{n=0}^{\infty}1$

both fail the Divergence Test

since $\lim_{n\to\infty}(-1)^n$ does not exist (so $\neq 0$)

and $\lim_{n\to\infty}1 = 1 \neq 0$

so the cases where $x=\pm 2$ cause the series to diverge.

Root Test

(Theoretically a bit stronger than the Ratio Test, but harder to use in most cases.)

Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

1) If $L<1$, the series $\sum_{n=0}^{\infty} a_n$ converges absolutely.

2) If $L>1$, the series $\sum_{n=0}^{\infty} a_n$ diverges.

3) If $L=1$, then the test is inconclusive.
($\sum_{n=0}^{\infty} a_n$ could converge absolutely, converge conditionally or diverge)

It's especially well adapted to situations where a_n is built out of n^{th} powers using only mult'n. & division.

Example: $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ Try the Root Test.

$\lim_{n\to\infty} \left(\frac{3^n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{3}{n} = 0<1$ so the series converges absolutely by the Root Test.