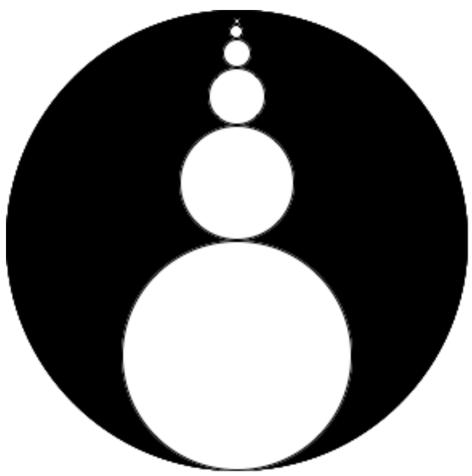
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2012

Solutions to Assignment #1
Designs for a (non-Olympic) diskus?!

Consider the shape obtained as follows:

0. Start with a disk of radius 1.
1. Remove a disk of radius $\frac{1}{2}$ that just touches the centre and the edge of the larger disk.
2. Remove a disk of radius $\frac{1}{4}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches the disk removed at step 1.
3. Remove a disk of radius $\frac{1}{8}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches the disk removed at step 2.
4. Remove a disk of radius $\frac{1}{16}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches the disk removed at step 3.
$\vdots$
n. Remove a disk of radius $\frac{1}{2^n}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches the disk removed at step $n-1$.
$\vdots$

The object obtained after the first few steps of this process is illustrated below:



1. Find a formula (in terms of n) for the area of the shape obtained at step n. *[4]*

Note: Just in case, the area of a circle of radius r is πr^2 ...

SOLUTION. At step 0 we have a disk of radius 1, whose area is therefore $\pi 1^2 = \pi$.

In step 1 we remove a disk of radius $\frac{1}{2}$, whose area is $\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$, leaving an object with area $\pi - \frac{\pi}{4} = \pi\left(1 - \frac{1}{4}\right)$.

In step 2 we remove a disk of radius $\frac{1}{4}$, whose area is $\pi\left(\frac{1}{4}\right)^2 = \frac{\pi}{16}$, leaving an object with area $\pi - \frac{\pi}{4} - \frac{\pi}{16} = \pi\left(1 - \frac{1}{4} - \frac{1}{16}\right)$.

In general, in step $n \geq 1$ we remove a disk of radius $\frac{1}{2^n}$, whose area is $\pi\left(\frac{1}{2^n}\right)^2 = \frac{\pi}{2^{2n}} = \frac{\pi}{4^n}$, leaving an object with area $\pi - \frac{\pi}{4} - \frac{\pi}{16} - \cdots - \frac{\pi}{4^n} = \pi\left(1 - \frac{1}{4} - \frac{1}{16} - \cdots - \frac{1}{4^n}\right)$.

To simplify the expression obtained above, we use the formula for the sum of a finite geometric series:

$$a + ar + ar^2 + ar^3 + ar^k = a\frac{1 - r^{k+1}}{1 - r}$$

In this case, we have

$$1 - \frac{1}{4} - \frac{1}{16} - \cdots - \frac{1}{4^n} = 1 - \left(\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n}\right),$$

and $\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n}$ is a finite geometric series with $a = \frac{1}{4}$, $r = \frac{1}{4}$, and $k = n - 1$. (Why is $k = n - 1$? If it's not obvious, think about it for bit ...) Thus

$$\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n} = \frac{1}{4} \cdot \frac{1 - \left(\frac{1}{4}\right)^{n-1+1}}{1 - \frac{1}{4}} = \frac{1}{4} \cdot \frac{1 - \left(\frac{1}{4}\right)^n}{\frac{3}{4}} = \frac{1 - \frac{1}{4^n}}{4 \cdot \frac{3}{4}} = \frac{1 - \frac{1}{4^n}}{3},$$

so the area of the shape obtained at step n of the process is:

$$\pi\left(1 - \frac{1 - \frac{1}{4^n}}{3}\right) = \pi\left(\frac{3}{3} - \frac{1 - \frac{1}{4^n}}{3}\right) = \pi\frac{2 + \frac{1}{4^n}}{3} = \frac{\pi}{3}\left(2 + \frac{1}{4^n}\right)$$

Whew! ■

2. What is the area of the shape obtained after infinitely many steps? *[1]*

SOLUTION. As n gets larger, 4^n gets larger (much faster!) without any sort of upper bound, so $\frac{1}{4^n}$ gets smaller and smaller, tending to 0. It follows that the area of the shape obtained after infinitely many steps is $\frac{\pi}{3}(2 + 0) = \frac{2\pi}{3}$. ■

Now consider the shape obtained as follows:

0. Start with a disk of radius 1.

1. Remove a disk of radius $\frac{1}{2}$ that just touches the centre and the edge of the larger disk.

2. Add back a disk of radius $\frac{1}{4}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches both previous disks.

3. Remove a disk of radius $\frac{1}{8}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches all the previous disks.

4. Add back a disk of radius $\frac{1}{16}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches all the previous disks.

⋮

2k+1. Remove a disk of radius $\frac{1}{2^{2k+1}}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches all the previous disks.

$\vdots$

2k+2. Add back a disk of radius $\frac{1}{2^{2k+2}}$ whose centre is on the straight line defined by the centres of the previous disks and which just touches all the previous disks.

$\vdots$

The object obtained after the first few steps of this process is illustrated below:

3. Find a formula (or formulas) for the area of the shape obtained at step n (or steps $2k+1$ and $2k+2$). *[4]*

SOLUTION. At step 0 we have a disk of radius 1, whose area is therefore $\pi 1^2 = \pi$.

In step 1 we remove a disk of radius $\frac{1}{2}$, whose area is $\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$, leaving an object with area $\pi - \frac{\pi}{4} = \pi\left(1 - \frac{1}{4}\right)$.

In step 2 we add a disk of radius $\frac{1}{4}$, whose area is $\pi\left(\frac{1}{4}\right)^2 = \frac{\pi}{16}$, leaving an object with area $\pi - \frac{\pi}{4} + \frac{\pi}{16} = \pi\left(1 - \frac{1}{4} + \frac{1}{16}\right)$.

In general, in step $n \geq 1$ we remove or add a disk of radius $\frac{1}{2^n}$, whose area is $\pi\left(\frac{1}{2^n}\right)^2 = \frac{\pi}{2^{2n}} = \frac{\pi}{4^n}$, depending on whether n is odd or even, repectively, giving us an object with area $\pi - \frac{\pi}{4} + \frac{\pi}{16} - \cdots + (-1)^n \frac{\pi}{4^n} = \pi\left(1 - \frac{1}{4} - \frac{1}{16} - \cdots + \left(\frac{-1}{4}\right)^n\right)$.

To simplify the expression obtained above, we use the formula for the sum of a finite geometric series:

$$a + ar + ar^2 + ar^3 + ar^k = a\frac{1 - r^{k+1}}{1 - r}$$

In this case, we have

$$1 - \frac{1}{4} + \frac{1}{16} - \cdots + \left(\frac{-1}{4}\right)^n,$$

which is a finite geometric series with $a = -1$, $r = -\frac{1}{4}$, and $k = n$. (Why is $k = n$ in this case and not $n - 1$ as in the solution to question **1**?) Thus

$$1 - \frac{1}{4} + \frac{1}{16} - \cdots + \left(\frac{-1}{4}\right)^n = \frac{1 - \left(-\frac{1}{4}\right)^{n+1}}{1 - \left(-\frac{1}{4}\right)} = \frac{1 - \left(-\frac{1}{4}\right)^{n+1}}{\frac{5}{4}} = \frac{4}{5}\left(1 - \left(-\frac{1}{4}\right)^{n+1}\right),$$

so the area of the shape obtained at step n of the process is:

$$\pi\frac{4}{5}\left(1-\left(-\frac{1}{4}\right)^{n+1}\right)=\frac{4\pi}{5}\left(1-\left(-\frac{1}{4}\right)^{n+1}\right),$$

Whew (again)! ■

4. What is the area of the shape obtained after infinitely many steps? *[1]*

Solution. As n gets larger, 4^n gets larger (much faster!) without any sort of upper bound, so $\left(-\frac{1}{4}\right)^{n+1} = (-1)^n\frac{1}{4^n}$ gets smaller and smaller in absolute value, tending to 0. Thus the area of the shape obtained after infinitely many steps is $\frac{4\pi}{5}(1-0)=\frac{4\pi}{5}$. ■