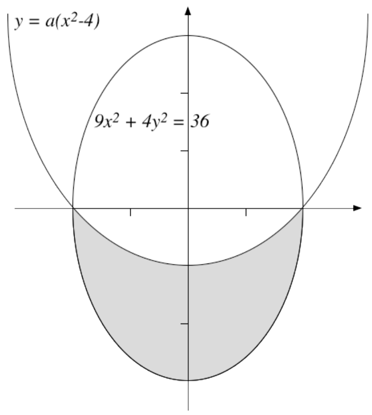
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

Solution to Assignment #8
Smile!?

The ellipse with equation $9x^2 + 4y^2 = 36$ (in standard form $\frac{x^2}{4} + \frac{y^2}{9} = 1$) has its x-intercepts at $x = \pm 2$. The parabola $y = a\left(x^2 - 4\right) = ax^2 - 4a$, where we require that $a > 0$, also has its x-intercepts at $x = \pm 2$.



1. Find the value of a so that the area of the part of the ellipse $9x^2 + 4y^2 = 36$ below the parabola $y = a\left(x^2 - 4\right)$ is exactly 2π. *[10]*

HINT: This is doable by hand – though you may have to read ahead to learn about trigonometric substitutions to do the relevant integral – but it would be a lot less work to use `Maple`...

NOTE: Not that you need to know it for this problem, but the area enclosed by the ellipse with equation $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ is πcd. In this case $c = 2$ and $d = 3$, which makes the area of the whole ellipse 6π, so the question asks you to find the value of a which makes the area of the region $\frac{1}{3}$ the area of the whole ellipse.

SOLUTION. We first set up the integral for the area, noting that $y = a\left(x^2 - 4\right)$ is above the lower part of the ellipse. The equation of the lower part of the ellipse is obtained as follows:

$$9x^2 + 4y^2 = 36 \implies y^2 = \frac{36 - 9x^2}{4} \implies y = -\sqrt{\frac{36 - 9x^2}{4}} = -3\sqrt{1 - \frac{x^2}{4}}$$

The area integral is then:

$$\int_{-2}^{2} \left(a\left(x^2 - 4\right) - \left(-3\sqrt{1 - \frac{x^2}{4}} \right) \right) dx = \int_{-2}^{2} \left(a\left(x^2 - 4\right) + 3\sqrt{1 - \frac{x^2}{4}} \right) dx$$

We thus need to solve for a in the equation:

$$\int_{-2}^{2}\left(a\left(x^2-4\right)+3\sqrt{1-\frac{x^2}{4}}\right)\,dx=2\pi$$

Rather than try to work the integral by hand and then solve for a, we let `Maple` do the heavy lifting:

```
> solve(int(a*(x^2-4)+3*sqrt(1-(1/4)*x^2),x=-2..2)=2*Pi,a);
```

$$\frac{3}{32}\pi$$

For those who prefer a decimal approximation:

```
> fsolve(int(a*(x^2-4)+3*sqrt(1-(1/4)*x^2),x=-2..2)=2*Pi,a);
```

$$0.2945243112$$

That's all folks! $\square$