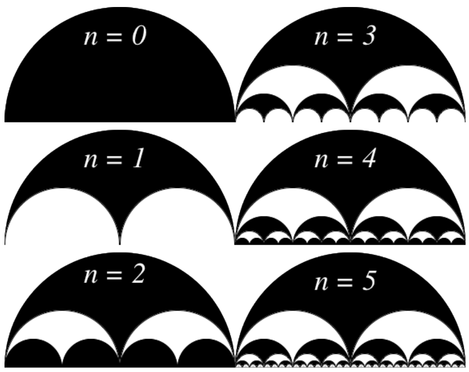
Mathematics 1100Y – Calculus I: Calculus of one variable
Trent University, Summer 2011

Solutions to Assignment #1
Alien Batman logo?!

Consider the shape obtained as follows:

0. Start with a half-disk of radius 1.
1. Remove two side-by-side half-disks of radius $\frac{1}{2}$ (straight edges aligned!).
2. Add back in four side-by-side half-disks of radius $\frac{1}{4}$ (straight edges aligned!).
3. Remove eight side-by-side half-disks of radius $\frac{1}{8}$ (straight edges aligned!).
4. Add back in sixteen side-by-side half-disks of radius $\frac{1}{16}$ (straight edges aligned!).
$\vdots$
2k. Add back in [how many?] side-by-side half-disks of radius [?] (straight edges aligned!).
2k+1. Remove [how many?] side-by-side half-disks of radius [?] (straight edges aligned!).
$\vdots$

The first few steps of this process are illustrated below:



1. How many half-disks are added back in or removed at step n of the process? What is their radius? *[5]*

Solution. At step 0 we add $1 = 2^0$ half-disk of radius $1 = 2^0$. At step 1 we remove $2 = 2^1$ half-disks, each of radius $\dfrac{1}{2} = \dfrac{1}{2^1}$. At step 2 we add $4 = 2^2$ half-disks, each of radius $\frac{1}{4} = \frac{1}{2^2}$. At step 3 we remove $8 = 2^3$ half-disks, each of radius $\frac{1}{8} = \frac{1}{2^3}$. At step 5 ...

It should be clear from this pattern that at step n one adds (if n is even) or removes (if n is odd) 2^n half-disks, each of radius $\dfrac{1}{2^n}$. $\square$

2. What is the area of the shape obtained after infinitely many steps of this process? *[5]*

Solution. The area of a half-disk of radius r is $\dfrac{\pi}{2}r^2$. Using the information we obtained in **1**, the area of the shape is therefore the infinite sum:

$$\frac{\pi}{2}1^2 - \frac{\pi}{2}2\left(\frac{1}{2}\right)^2 + \frac{\pi}{2}4\left(\frac{1}{4}\right)^2 - \frac{\pi}{2}8\left(\frac{1}{8}\right)^2 + \cdots = \frac{\pi}{2}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\right)$$

It remains to determine what this sum amounts to. To make this a bit easier, we will work with $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ and multiply by $\frac{\pi}{2}$ later.

One way to find the sum is to simply look at the partial sums and see where their values are headed.

n	*Partial sum to nth term*	*Decimal value*
0	1	1.0
1	$1-\frac{1}{2}$	0.5
2	$1-\frac{1}{2}+\frac{1}{4}$	0.75
3	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$	0.625
4	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$	0.6875
5	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$	0.65625
6	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}$	0.671875
7	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}$	0.6640625
8	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}$	0.66796875
9	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\frac{1}{512}$	0.666015625
10	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\frac{1}{512}+\frac{1}{1024}$	0.6669921875
11	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\frac{1}{512}+\frac{1}{1024}-\frac{1}{2048}$	0.66650390625
$\vdots$	$\vdots$	$\vdots$

Looking at the decimal values carefully, it is not hard to see that as n increases, the partial sums alternately hop over and under $0.6666666\cdots = \frac{2}{3}$, getting ever closer as the hops decrease in size. The sum of the full infinite series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ should therefore be $\frac{2}{3}$.

Another way to find the sum is to observe that $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$ is a *geometric series*, that is, one of the form $a+ar+ar^2+ar^3+\cdots$. A little looking up tells us that as long as the *common ratio* between successive terms, r, has absolute value less than 1, a geometric series sums to $\frac{a}{1-r}$. In our case $a = 1$ and $r = -\frac{1}{2}$, so $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$.

Either way, it follows that the area of the shape in question is

$$\frac{\pi}{2}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\right) = \frac{\pi}{2} \cdot \frac{2}{3} = \frac{\pi}{3}. \quad \square$$