Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010
Solutions to Test 1

1. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Find the slope of the tangent line to $y = \tan(x)$ at $x = 0$.

SOLUTION. The slope of the tangent line at a given point is given by evaluating the derivative at the given point. In this case, $\frac{dy}{dx} = \frac{d}{dx}\tan(x) = \sec^2(x)$. At $x = 0$ this gives $\sec^2(0) = \frac{1}{\cos^2(0)} = \frac{1}{1} = 1$, so the tangent line to $y = \tan(x)$ at $x = 0$ has slope 1. ■

b. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = x^2$.

SOLUTION. Here goes:

$$\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h \to 0} \frac{(1+h)^2 - 1^1}{h} \\
&= \lim_{h \to 0} \frac{1^1 + 2 \cdot 1 \cdot h + h^2 - 1^2}{h} \\
&= \lim_{h \to 0} \frac{2h + h^2}{h} \\
&= \lim_{h \to 0} (2 + h) = 2 + 0 = 2 \qquad ■
\end{aligned}$$

c. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 1} (2x - 1) = 1$.

SOLUTION. We need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - 1| < \delta$, then $|(2x - 1) - 1| < \varepsilon$. Given a $\varepsilon > 0$, we reverse-engineer the $\delta > 0$ we need:

$$\begin{aligned}
|(2x - 1) - 1| < \varepsilon &\iff |2x - 2| < \varepsilon \\
&\iff |2(x - 1)| < \varepsilon \\
&\iff 2\,|x - 1| < \varepsilon \\
&\iff |x - 1| < \frac{\varepsilon}{2}
\end{aligned}$$

Since each step above is reversible, it follows that that if $\delta = \frac{\varepsilon}{2}$, then $|(2x - 1) - 1| < \varepsilon$ whenever $|x - 1| < \delta = \frac{\varepsilon}{2}$. Thus $\lim_{x \to 1} (2x - 1) = 1$ by the $\varepsilon - \delta$ definition of limits. ■

2. Find $\frac{dy}{dx}$ in any *three* (3) of **a–d**. *[9 = 3 × 3 each]*

a. $y = \dfrac{x}{x+1}$

Solution. Apply the Quotient Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{\left[\frac{d}{dx}x\right](x+1) - x\left[\frac{d}{dx}(x+1)\right]}{(x+)^2} = \frac{1(x+1) - x1}{(x+)^2} = \frac{1}{(x+)^2} \quad \blacksquare$$

b. $x^2 + y^2 = 4$

Solution i. Use implicit differentiation and the Chain Rule:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}4 \Longrightarrow \frac{d}{dx}x^2 + \frac{d}{dx}t^2 = 0 \Longrightarrow 2x + \left(\frac{d}{dy}y^2\right)\frac{dy}{dx} = 0$$
$$\Longrightarrow 2x + 2y\frac{dy}{dx} = 0 \Longrightarrow 2y\frac{dy}{dx} = -2x \Longrightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y} \quad \blacksquare$$

Solution ii. Solve for y and then differentiate using the Chain Rule. First,

$$x^2 + y^2 = 4 \Longrightarrow y^2 = 4 - x^2 \Longrightarrow y = \pm\sqrt{(4-x^2)}\,.$$

Second,

$$\frac{dy}{dx} = \frac{d}{dx} \pm \sqrt{(4-x^2)} = \frac{1}{\pm 2\sqrt{(4-x^2)}} \cdot \frac{d}{dx}\left(4-x^2\right) = \frac{1}{\pm 2\sqrt{(4-x^2)}} \cdot (0-2x)$$
$$= \frac{-2x}{\pm 2\sqrt{(4-x^2)}} = \frac{-x}{\pm\sqrt{(4-x^2)}} = -\frac{x}{y}\,. \quad \blacksquare$$

c. $y = \displaystyle\int_0^x t\cos(3t)\,dt$

Solution. By the Fundamental Theorem of Calculus:

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x t\cos(3t)\,dt = x\cos(3t) \quad \blacksquare$$

d. $y = \ln\left(x^3\right)$

Solution i. Simplify, then differentiate. First, $y = \ln\left(x^3\right) = 3\ln(x)$. Second,

$$\frac{dy}{dx} = \frac{d}{dx}3\ln(x) = 3 \cdot \frac{1}{x} = \frac{3}{x}\,. \quad \blacksquare$$

SOLUTION II. Differentiate using the Chain Rule, then simplify:

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(x^3\right) = \frac{1}{x^3}\cdot\frac{d}{dx}x^3 = \frac{1}{x^3}\cdot 3x^2 = \frac{3}{x} \quad \blacksquare$$

3. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Explain why $\lim\limits_{x\to 0}\dfrac{x}{|x|}$ doesn't exist.

SOLUTION. Note that when $x > 0$, $|x| = x$, so $\dfrac{x}{|x|} = 1$, and when $x < 0$, $x = -|x|$, so $\dfrac{x}{|x|} = -1$. It follows that $\lim\limits_{x\to 0^-}\dfrac{x}{|x|} = \lim\limits_{x\to 0^-} -1 = -1$ and $\lim\limits_{x\to 0^+}\dfrac{x}{|x|} = \lim\limits_{x\to 0^+} 1 = 1$, so $\lim\limits_{x\to 0}\dfrac{x}{|x|}$ can't exist since $-1 \neq 1$. $\blacksquare$

b. A spherical balloon is being inflated at a rate of 1 m^3/s. How is its radius changing at the instant that it is equal to 2 m? [The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.]

SOLUTION. On the one hand, we are given that $\dfrac{dV}{dt} = 1$; on the other hand, using the Chain Rule,

$$\frac{dV}{dt} = \frac{d}{dt}\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{d}{dr}r^3\right)\frac{dr}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}\,.$$

It follows that $1 = \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$, so $\dfrac{dr}{dt} = \dfrac{1}{4\pi r^2}$. Thus, at the instant that $r = 2$ m, we have $\dfrac{dr}{dt} = \dfrac{1}{4\pi 2^2} = \dfrac{1}{16\pi}$ m/s. $\blacksquare$

c. Use the Left-Hand Rule to find $\displaystyle\int_1^3 x\,dx$. $\left[\sum_{i=0}^{n-1} i = 0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2}\right]$

SOLUTION. Not letting the right hand know what the left hand is doing:

$$\int_1^3 x\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{3-1}{n}\cdot\left(1 + i\frac{3-1}{n}\right) \qquad \text{[Since our function is just } f(x) = x.\text{]}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{2}{n}\left(1 + i\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\sum_{i=0}^{n-1}\left(1 + i\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left(\left[\sum_{i=0}^{n-1}1\right] + \left[\sum_{i=0}^{n-1}i\frac{2}{n}\right]\right) = \lim_{n\to\infty}\frac{2}{n}\left(n + \left[\frac{2}{n}\sum_{i=0}^{n-1}i\right]\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left(n + \frac{2}{n}\cdot\frac{n(n-1)}{2}i\right) = \lim_{n\to\infty}\frac{2}{n}\left(n + (n-1)\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}(2n-1) = \lim_{n\to\infty}\left(4 - \frac{2}{n}\right) = 4 - 0 = 4 \qquad \blacksquare$$

4. Let $f(x) = \dfrac{x^2}{x^2+1}$. Find the domain and all the intercepts, vertical and horizontal asymptotes, and maxima and minima of $f(x)$, and sketch its graph using this information. *[11]*

SOLUTION. We run through the checklist:

i. Domain. $f(x) = \dfrac{x^2}{x^2+1}$ always makes sense because the denominator $x^2+1 \geq 1 > 0$ for all x. Thus the domain of $f(x)$ is all of $\mathbb{R}$; note that $f(x)$ must also be continuous everywhere. $\square$

ii. Intercepts. $f(0) = 0$, so $(0,0)$ is the y-intercept. Since $f(x) = \dfrac{x^2}{x^2+1} = 0$ is only possible when the numerator is 0, any x-intercepts occur when $x^2 = 0$, *i.e.* when $x = 0$. Thus $(0,0)$ is the only x-intercept, as well as the y-intercept. $\square$

iii. Vertical asymptotes. Since $f(x)$ is defined and continuous on all of $\mathbb{R}$ it has no vertical asymptotes. (As noted in *i* above, this is because the denominator is never 0.) $\square$

iv. Horizontal asymptotes. We check how $f(x)$ behaves as $x \to \pm\infty$:

$$\lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{x^2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0^+} = 1^-$$
$$\lim_{x\to-\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0+)} = 1^-$$

Thus $f(x)$ has $x = 1$ as a horizontal asymptote in both directions. Note that because $\frac{x^2}{x^2+1} = \frac{1}{1+1/x^2} < 1$ for all x, $f(x)$ approaches this asymptote from below in both directions. $\square$

v. Maxima and minima. Since $f(x)$ is defined and continuous on all of $\mathbb{R}$, we only have to check any critical points to find any local maxima and minima. We first compute the derivative:

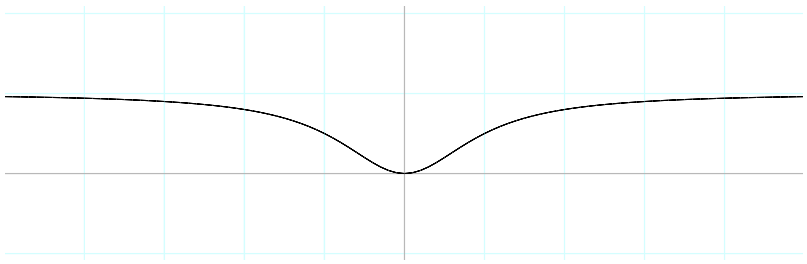
$$\begin{aligned} f'(x) = \frac{d}{dx}\left(\frac{x^2}{x^2+1}\right) &= \frac{\left[\frac{d}{dx}x^2\right]\left(x^2+1\right) - x^2\left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^2} \\ &= \frac{2x\left(x^2+1\right) - x^2\left(2x+0\right)}{\left(x^2+1\right)^2} \\ &= \frac{2x^3+2x-2x^3}{\left(x^2+1\right)^2} = \frac{2x}{\left(x^2+1\right)^2} \end{aligned}$$

Since the denominator is never 0, $f'(x)$ is defined for all x and $f'(x) = 0$ only when the numerator, $2x$, is 0, *i.e.* when $x = 0$. Thus $x = 0$ is the only critical point. From the behaviour around the critical point,

$$\begin{array}{cccc} x & (-\infty,0) & 0 & (0,\infty) \\ f'(x) & <0 & 0 & >0 \\ f(x) & \downarrow & 0 & \uparrow \end{array},$$

$f(0) = 0$ is a local (and absolute!) minimum. Note that $f(x)$ has no local maxima. $\square$

vi. Graph.



This graph was drawn using a program called `EdenGraph`. □

Whew! ■

Bonus. Find any inflection points of $f(x) = \dfrac{x^2}{x^2+1}$ as well. *[3]*

SOLUTION. We add one more item to the checklist above:

vii. Inflection points. Note that $f'(x)$ is defined and differentiable for all x. We first compute the second derivative:

$$\begin{aligned}
f''(x) &= \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{2x}{\left(x^2+1\right)^2}\right) = \frac{\left[\frac{d}{dx}2x\right]\left(x^2+1\right)^2 - 2x\left[\frac{d}{dx}\left(x^2+1\right)^2\right]}{\left(\left(x^2+1\right)^2\right)^2} \\
&= \frac{2\left(x^2+1\right)^2 - 2x\left[2\left(x^2+1\right)\cdot\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^4} \\
&= \frac{2\left(x^2+1\right)^2 - 2x\left[2\left(x^2+1\right)\cdot(2x+0)\right]}{\left(x^2+1\right)^4} = \frac{2\left(x^2+1\right)^2 - 2x\left[4x\left(x^2+1\right)\right]}{\left(x^2+1\right)^4} \\
&= \frac{2\left(x^2+1\right)^2 - 8x^2\left(x^2+1\right)}{\left(x^2+1\right)^4} = \frac{2\left(x^2+1\right) - 8x^2}{\left(x^2+1\right)^3} = \frac{2-6x^2}{\left(x^2+1\right)^3}
\end{aligned}$$

Since the denominator is never 0, $f''(x)$ is defined for all x and $f''(x) = 0$ only when the numerator, $2 - 6x^2$, is 0, *i.e.* when $x = \pm\dfrac{1}{\sqrt{3}}$. Thus the potential inflection points of $f(x)$ are $x = -\dfrac{1}{\sqrt{3}}$ and $x = \dfrac{1}{\sqrt{3}}$. From the behaviour around these points,

$$\begin{array}{cccccc}
x & \left(-\infty, -\frac{1}{\sqrt{3}}\right) & -\frac{1}{\sqrt{3}} & \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{3}}, \infty\right) \\
f''(x) & < 0 & 0 & > 0 & 0 & < 0 \\
f'(x) & \downarrow & & \uparrow & & \downarrow \\
f(x) & \text{concave down} & \frac{1}{4} & \text{concave up} & \frac{1}{4} & \text{concave down}
\end{array},$$

it follows that $f(x)$ has two inflection points, at $x = -\dfrac{1}{\sqrt{3}}$ and $x = \dfrac{1}{\sqrt{3}}$. □

Bonus whew! ■