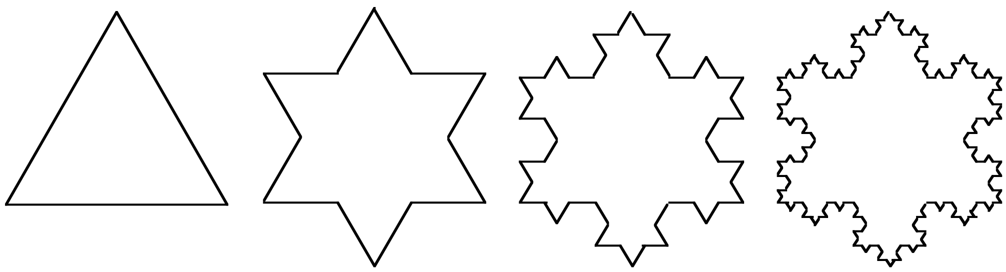
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Assignment #1
The Snowflake Curve*
Due on Wednesday, 19 May, 2010.

Suppose one has an equilateral triangle with sides of length 1. If one modifies each of the line segments composing the triangle by cutting out the middle third of the segment, and then inserting an outward-pointing "tooth," both of whose sides are as long as the removed third, one gets a six-pointed star. Suppose one repeats this process for each of the line segments making up the star, then to each of the line segments making up the resulting figure, and so on, as in the diagram:



Note that the lengths of the line segments at each stage are a third of the length of the segments at the preceding stage. For the sake of being definite, let's say we have the triangle at step 0 of the process, the six-pointed star at step 1 of the process, the next shape at step 2 of the process, and so on. The curve which is the limit of this process, if one takes infinitely many steps, is often called the *snowflake curve**. We will try to discover the length of this curve and the area of the region that it encloses below.

1. What are the lengths of the curves we have at steps 0, 1, 2, and 3 of the process? *[1]*

SOLUTION I. Note that at each step we modify each existing line segment by replacing the middle third of the segment with an outward-pointing tooth both of whose sides are as long as the removed third. Since the other two thirds are not affected at this step, the total length of the modified segment is $1 + \frac{1}{3} = \frac{4}{3}$ of the length of the old segment. Thus the total length increases by $\frac{4}{3}$ at each step.

At step 0, we have an equilateral triangle with three sides of length 1, for a total length of $3 = 3 \times 1$.

At step 1, we then get a total length of $4 = \frac{4}{3} \times 3$.

At step 2, we then get a total length of $\frac{16}{3} = \frac{4}{3} \times 4 = \left(\frac{4}{3}\right)^2 \times 3$.

At step 3, we then get a total length of $\frac{64}{9} = \frac{4}{3} \times \frac{16}{3} = \left(\frac{4}{3}\right)^3 \times 3$. ■

* Also known as the *Koch curve.*

Here is an alternate, slightly more complicated, solution whose approach will be helpful in answering some of the later questions.

SOLUTION II. First, note that the number of line segments (disregarding for the moment their length) in the perimeter of the shape at step $n+1$ is 4 times the number of line segments in the perimeter of the shape at step n. Since we have 3 line segments at step 0, this means we have 12 at step 1, 48 at step 2, and so on. In general, we have $3 \cdot 4^n$ line segments at step n.

Second, note that at each stage the length of each individual line segments (disregarding for the moment how many there are) in the perimeter of the shape at step $n+1$ is $\frac{1}{3}$ times the length of each individual line segment in the perimeter of the shape at step n. Since the line segments at step 1 each have length 1, this means that each line segment at step 1 has length $\frac{1}{3}$, each at step 2 has length $\frac{1}{9}$, and so on. In general, each line segment in the perimeter of the shape at step n has length $\frac{1}{3^n}$.

It follows that the total length of the perimeter of the shape at step n is $3 \cdot 4^n \cdot \frac{1}{3^n} = 3 \cdot \frac{4^n}{3^n} = 3\left(\frac{4}{3}\right)^n$. [Note that this answers **2** while we're at it!] Hence the length of the curve at step 0 is 3, at step 1 is 4, at step 2 is $\frac{16}{3}$, and at step 3 is $\frac{64}{9}$. ■

2. What is the length of the curve at step n of the process? *[1]*

SOLUTION. From the pattern observed in the solution to **1**, the total length at step n should be 3 times $\frac{4}{3}$ at each step, *i.e.* $3\left(\frac{4}{3}\right)^n$. ■

3. Suppose $L > 0$ is some positive number. Find an number N such that if $n \geq N$, then the length of the curve at step n is $\geq L$. *[2]*

SOLUTION. Using the solution to **2**, this boils down to finding out how big n has to be to ensure that $3\left(\frac{4}{3}\right)^n \geq L$.

$$\begin{aligned}
3\left(\frac{4}{3}\right)^n \geq L &\iff \left(\frac{4}{3}\right)^n \geq \frac{L}{3} \\
&\iff \ln\left(\left(\frac{4}{3}\right)^n\right) \geq \ln\left(\frac{L}{3}\right) \\
&\iff n\ln\left(\frac{4}{3}\right) \geq \ln\left(\frac{L}{3}\right) \\
&\iff n \geq \frac{\ln\left(\frac{L}{3}\right)}{\ln\left(\frac{4}{3}\right)}
\end{aligned}$$

It follows that making $N = \left\lceil \frac{\ln\left(\frac{L}{3}\right)}{\ln\left(\frac{4}{3}\right)} \right\rceil$, *i.e.* the greatest integer $\geq \frac{\ln\left(\frac{L}{3}\right)}{\ln\left(\frac{4}{3}\right)}$, does the job.

Note that we're implicitly using the fact that $\ln(x)$ is an increasing function. ■

4. What does **3** suggest about the length of the snowflake curve? *[1]*

SOLUTION. It follows from the solution to **3** that no matter what number L we might pick for the length, the length at some stage gets above L and stays above it for all later stages. Thus the length of the snowflake curve, which is the limit of all the stages, must be infinite. ■

5. What are the areas of the shapes we have at steps 0, 1, 2, and 3 of the process? *[1]*

Solution. At step 0 we have an equilateral triangle with sides of length 1. I'll leave it to you to check that this has area $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$.

In step 1 we add the area of three smaller triangles, each with sides of length $\frac{1}{3}$. The area of each smaller triangle is then $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ of the area of the original triangle, *i.e.* $\frac{1}{9} \cdot \frac{\sqrt{3}}{4}$. Since there are three of them, this means the total area added at step 1 is $3 \cdot \frac{1}{9} \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^0 \cdot \frac{1}{3} \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12}$. Thus the area of the six-pointed star at step 1 is $\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12}$. (Why are we bothering with $\left(\frac{4}{9}\right)^0 = 1$ in this expression? Because that's what fits the pattern that develops below.)

In step 2 we add the area of $4 \times 3 = 12$ even smaller triangles, each with sides of length $\frac{1}{9} = \frac{1}{3} \times \frac{1}{3} = \left(\frac{1}{3}\right)^2$. (Why 12 of them? Because we stick one triangular tooth in the middle of each line segment we have after step 1, of which there are 4 for each side of the original triangle from step 0. See Solution ii to question **1**.) Each thus has area $\left(\frac{1}{9}\right)^2 = \frac{1}{81}$ of the area of the original triangle, *i.e.* $\left(\frac{1}{9}\right)^2 \cdot \frac{\sqrt{3}}{4}$, for a total added area of $12 \cdot \left(\frac{1}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12}$. Thus the area of the shape at step 2 is $\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12}$.

In step 3 we add the area of $4 \times 12 = 48$ even smaller triangles, each with sides of length $\frac{1}{27} = \left(\frac{1}{3}\right)^3$. (Why 48 of them? Because we stick one triangular tooth in the middle of each line segment we have after step 2, of which there are 4 for each side of the shape from step 1. See Solution ii to question **1**.) Each thus has area $\left(\frac{1}{27}\right)^2 = \left(\frac{1}{3}\right)^6 = \left(\frac{1}{9}\right)^3$ of the area of the original triangle, *i.e.* $\left(\frac{1}{9}\right)^3 \cdot \frac{\sqrt{3}}{4}$, for a total added area of $48 \cdot \left(\frac{1}{9}\right)^3 \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{12}$. Thus the area of the shape at step 3 is $\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{12}$.

■

6. What is the area of the shape at step n of the process? *[1]*

Solution. Using Solution ii to question **1**, at step $n-1$ of the process we have $3 \cdot 4^{n-1}$ line segments, and at step n of the process each line segment has length $\frac{1}{3^n}$, so each little triangle added at step n has area $\left(\frac{1}{3^n}\right)^2 = \frac{1}{3^{2n}} = \frac{1}{9^n}$ of the area of the original triangle, *i.e.* $\frac{1}{9^n} \cdot \frac{\sqrt{3}}{4}$. Hence the area *added* to the shape at step $n \geq 1$ is:

$$\begin{aligned}
&(\#\text{ triangles added at step } n) \times (\text{area of each triangle}) \\
&= (\#\text{ line segments at step } n-1) \times (\text{length of side})^2 \cdot (\text{area of original triangle}) \\
&= 3 \cdot 4^{n-1} \times \left(\left(\frac{1}{3}\right)^n\right)^2 \cdot \frac{\sqrt{3}}{4} = 3 \cdot \frac{4^n}{4} \times \frac{1}{9^4} \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^n \cdot \frac{3\sqrt{3}}{16} = \left(\frac{4}{9}\right)^{n-1} \cdot \frac{\sqrt{3}}{12}
\end{aligned}$$

It follows that the total area of the shape at step n is:

$$\begin{aligned}
&\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12} + \cdots + \left(\frac{4}{9}\right)^{n-1} \cdot \frac{\sqrt{3}}{12} \\
=&\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right) \qquad \blacksquare
\end{aligned}$$

7. What is the difference between the area of the shape we have at step n and $\frac{3\sqrt{3}}{4}$? *[2]*

SOLUTION. The difference between area of the shape we have at step n and $\frac{3\sqrt{3}}{4}$ is, of course:

$$\begin{aligned}&\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right) - \frac{3\sqrt{3}}{4} \\ =&\frac{\sqrt{3}}{12} \cdot \left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right) - \frac{\sqrt{3}}{2}\end{aligned}$$

This isn't very interesting in itself. [Unfortunately, that's an artifact of of your instructor making an error, really, in picking the value $\frac{3\sqrt{3}}{4}$. What should he have picked if he'd really been on the ball?] However, one can make it more interesting by observing that

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1} = \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}} = \frac{9}{5}\left(1 - \left(\frac{4}{9}\right)^n\right),$$

using the formula for the sum of a finite geometric series (see Example 1 in §11.2 of the text), which is always a little less than $\frac{9}{5}$. It follows that the difference we want is bounded above by:

$$\frac{\sqrt{3}}{12} \cdot \frac{9}{5} - \frac{\sqrt{3}}{2} = \frac{27\sqrt{3}}{60} - \frac{30\sqrt{3}}{60} = -\frac{3\sqrt{3}}{60} = -\frac{\sqrt{3}}{20} \qquad \blacksquare$$

8. What does your answer to **7** suggest about the area of the region enclosed by the snowflake curve? *[1]*

SOLUTION. The answer to **7** suggests that the area of the region enclosed by the snowflake curve is finite. In fact, it is given by

$$\lim_{n\to\infty} \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \frac{9}{5}\left(1 - \left(\frac{4}{9}\right)^n\right)\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \frac{9}{5} = \frac{15\sqrt{3}}{60} + \frac{9\sqrt{3}}{60} = \frac{24\sqrt{3}}{60} = \frac{2\sqrt{3}}{5},$$

since $\left(\frac{4}{9}\right)^n \to 0$ as $n \to \infty$. [This is the value your instructor should have used in question **7**.] $\blacksquare$