Fall25-1110-Lab-6

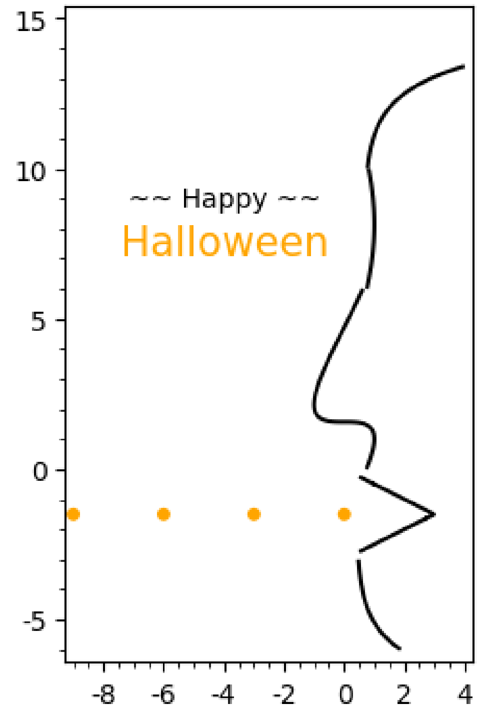
December 30, 2025

[40]:
```
# remember to include these two lines of code at the start of your document!
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = "all"
```

1 MATH1110 Lab 6: Critical and Inflection Points

Good to see you again!

[51]:
```
x, y = var('x, y')
one = implicit_plot(cos(x-y)==x, (x, -2, 4), (y, 0, 6), color = 'black')
two = implicit_plot(sin(-y/3-2)==x, (x, 0, 4), (y, 6, 10), color = 'black')
three = implicit_plot(log(9*x-6)+10==y, (x, 0, 4), (y, 10, 15), color = 'black')
four = implicit_plot(-abs(2*y+3)+3==x, (x, 1/2, 3), (y, -4, 3), color = 'black')
five = implicit_plot(-ln(5*x-2)-4==y, (x, 1/5, 3), (y, -6, -3), color = 'black')
six = points([[0, -3/2], [-3, -3/2], [-6, -3/2], [-9, -3/2]], size = 25,\
            color = 'orange')
seven = text('~~ Happy ~~', (-4, 9), color = 'black', fontsize = 10 )
eight = text('Halloween', (-4, 7.5), color = 'orange', fontsize = 15 )
show(one+two+three+four+five+six+seven+eight)
```



Objectives for today:

1. What have we done with Sage so far?
2. Critical points
3. Inflection points

1.1 1. What we've done so far

As a refersher, here's what we've done with Sage so far.

- Defining variables, building different kinds of functions
- Plotting, layering plots, making plots fancy
- Implicitly defined plots
- Limits
- Different uses of the *solve* command
- Derivatives
- Differential equations, initial value problems
- Identifying and solving errors

If you're feeling really lost on any of this content, please reach out to Maya or Stefan either in office hours or by email (sbilaniuk@trentu.ca, mapeters@trentu.ca).

1.2 2. Critical points

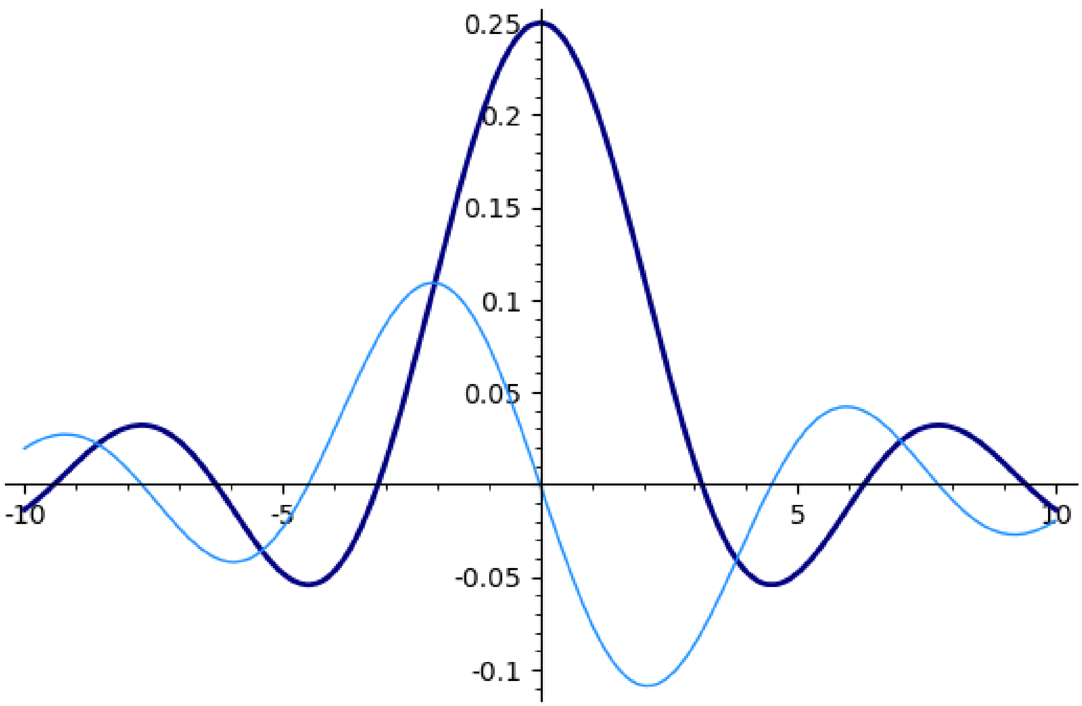
What is a critical point?

Critical points include maximums and minimums - places where the slope of our function switches from negative to positive or positive to negative. That is where the derivative of a function equals zero.

[15]:
```
var('t')
q = sin(t)/(4*t)
func = plot( q, t, -10, 10, color = 'navy', thickness = 2)
dfunc = plot( diff(q, t), t, -10, 10, color = 'dodgerblue')
func+dfunc
```

[15]: t

[15]:

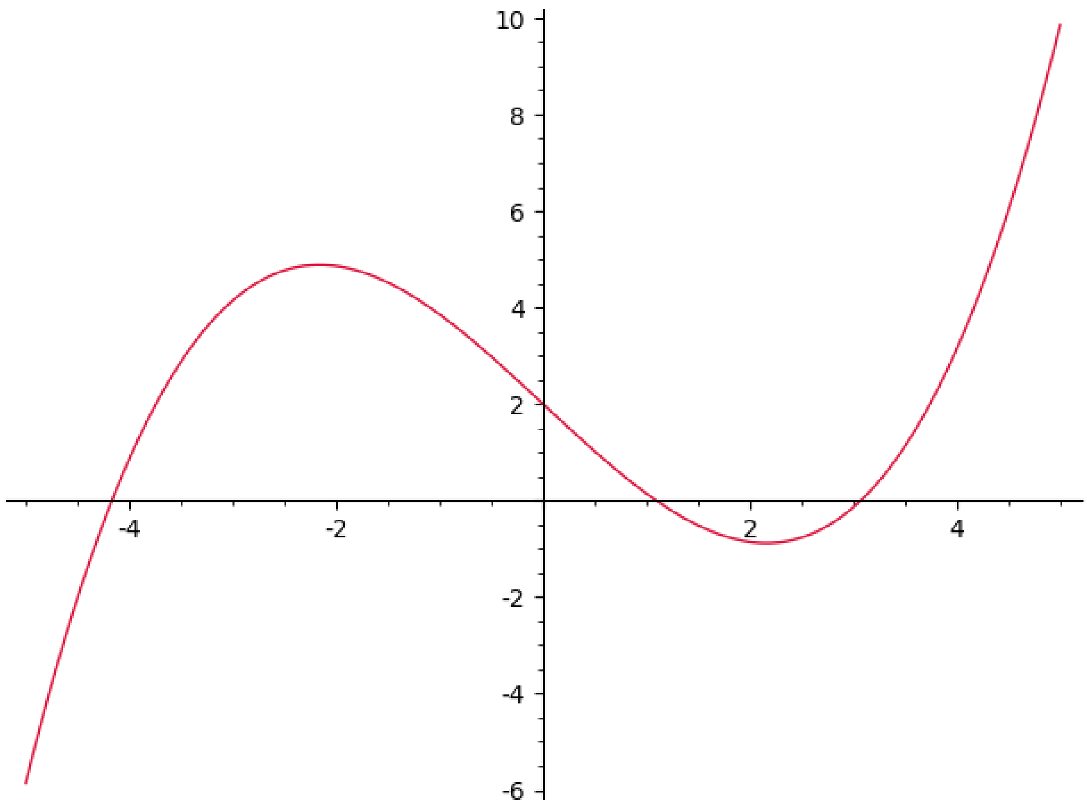


In the visualization above, we can see the navy blue maximum and minimum points correspond with where the lighter blue derivative function crosses the x-axis. What does the derivative being above or below the x-axis tell us about the original navy blue function?

Let's take the cubic function $y = \frac{x^3}{7} - 2x + 2$. We can use the *solve(_)* command to locate these critical points by creating an equation where our first derivative equal to zero. In the second argument, we are solving for x.

[1]:
```
x=var('x')
y=x^3/7-2*x+2
plot(y, x, -5, 5, color = 'crimson')
```

[1]:



Sage has another helpful command we can use to identify critical points. A "root" is a spot where a function passes across the x-axis. The command *find_root(_)* requires the function (not equation) we want the root of, and then the lower and upper bound of a domain within which we expect to find a root. Before using this function, it's very helpful to have a plot of your function to know where you're looking.

Plot $y(x)$ with its quadratic derivative after finding the roots of the derivative function. Notice how $y(x)$ is increasing whenever $y'(x)$ is positive and decreasing when $y'(x)$ is negative.

[ ]:

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1.2.1 Exercise question

1. Plot $g(x) = 2^{\left(\frac{x}{3}\right)} \sin(x)$ with its derivative on the domain [-5, 3]. Find all the local maximums and minimums of the function within this region.

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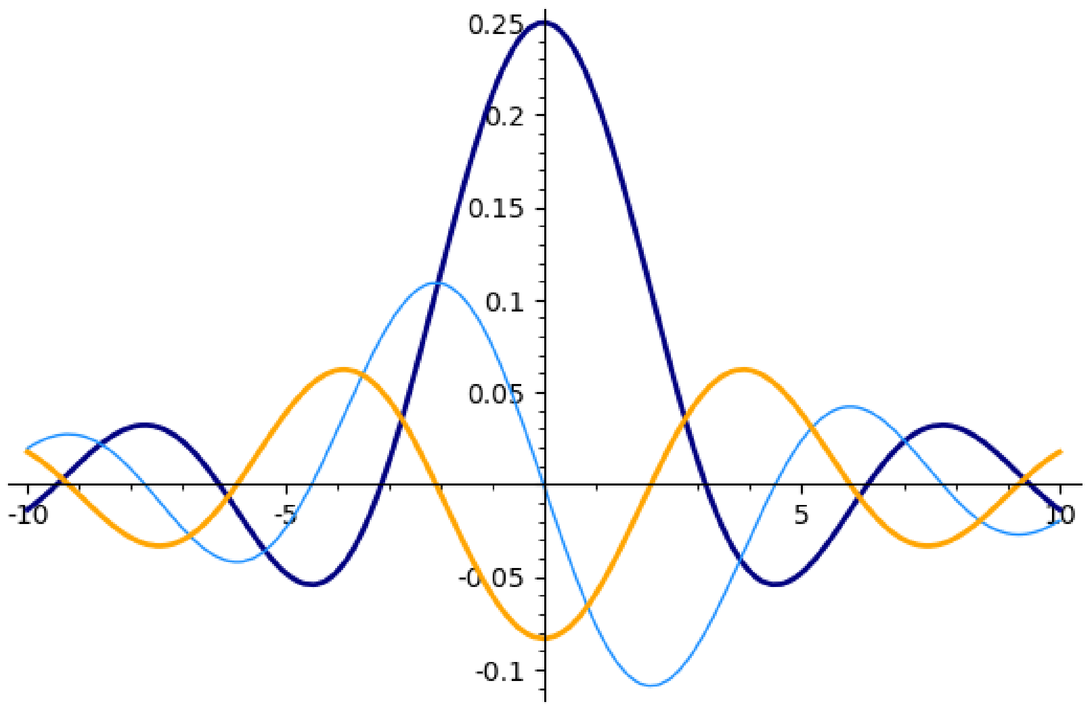
1.3 3. Inflection points

While the first derivative pertains to slope, the second derivative informs us on *concavity*. Concavity can be remembered as whether the curve is in a frowning ("concave down") or smiling ("concave up") arc, and the points of transition between the two are called infection points.

Some observations from the graph below: * The orange curve (second derivative) crossing the x-axis shows when we have a change in concavity in the original function. * The navy blue curve is concave down in the same regions where the orange curve is negative. * The navy blue curve is concave up in the same regions where the orange curve is positive. * Maximums and minimums of the light blue curve (first derivative) are where the orange curve crosses the x-axis.

[2]:
```
clear_vars()
var('t')
q = sin(t)/(4*t)
func = plot( q, t, -10, 10, color = 'navy', thickness = 2)
dfunc = plot( diff(q, t), t, -10, 10, color = 'dodgerblue')
d2func = plot( diff(q, t, 2), t, -10, 10, color = 'orange', thickness = 2)
func+dfunc+d2func
```

[2]:



If a function is smooth and continuous (differentiable across its entire domain), there will be an inflection point between every max and min.

Returning to our cubic example, show the second derivative. How many critical points will there be and how do you know? Use Sage to find the coordinates of the inflection point(s). If using the *find_root(__)* method, remember to plot the function first so you know subregion of the domain where we are looking for an inflection point.

[ ]:
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How are we doing? Is this making sense?

1.3.1 Exercise question

2. Find all inflection points of $g(x) = 2^{\left(\frac{x}{3}\right)} \sin{(x)}$ (same function as the previous example question) and plot g with the first and second derivative. Discuss in comments in your code how the increasing/decreasing behaviour and concavity of the original function are shown by the first and second derivative curves.

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