

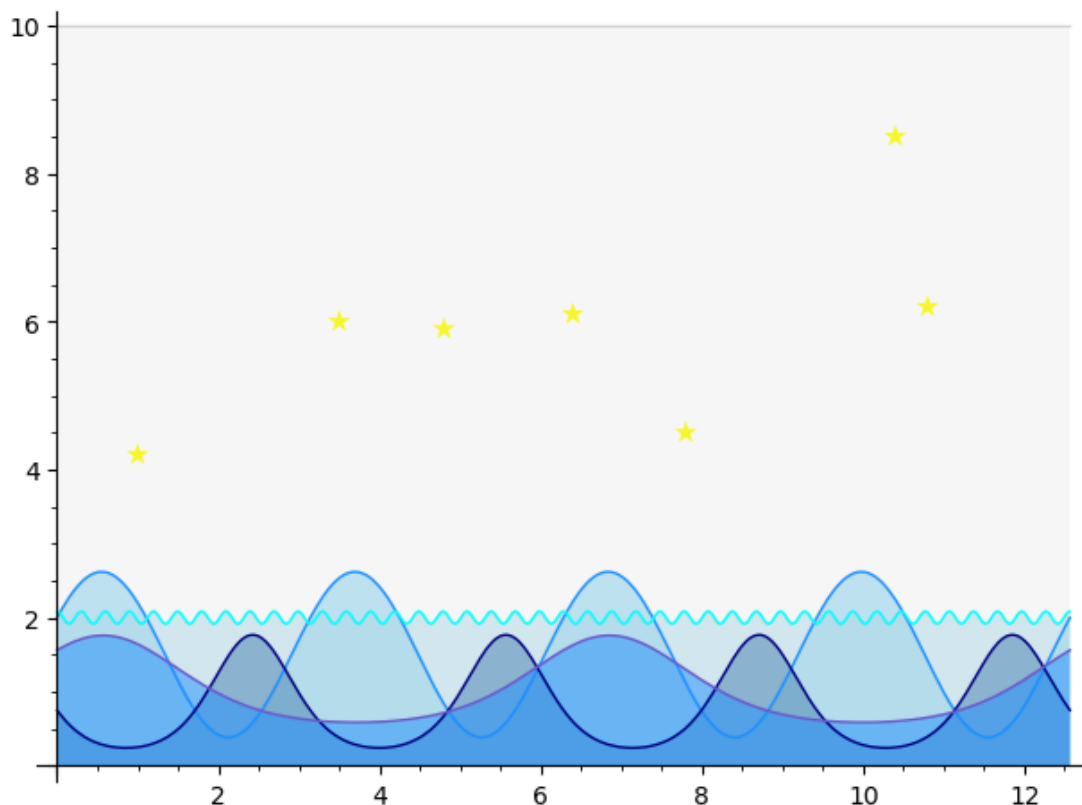
# Fall25-1110-Lab-5

December 30, 2025

```
[42]: # remember to include these two lines of code at the start of your document!
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = "all"
```

## 1 MATH1110 Lab 5: Differential Equations

```
[1]: x, y = var('x, y')
backdrop = plot(10, x, 0, 4*pi, color = 'lightgrey', fill = True, fillcolor = 'lightgrey', fillalpha = 0.2)
p1 = plot(cos(x)^2+sin(2*x)+1, x, 0, 4*pi, fill = True, fillcolor = 'skyblue', color = 'dodgerblue')
p2 = plot(cos(21*x)/12+2, x, 0, 4*pi, fill = True, fillcolor = 'lightblue', color = 'aqua')
p3 = plot(1.3*e^sin(2*x+3)/2, 0, 4*pi, fill = True, fillcolor = 'steelblue', color = 'navy')
p4 = plot(e^sin(x+1)/2+0.4, 0, 4*pi, fill = True, fillcolor = 'dodgerblue', color = 'slateblue')
pts = points( [[1, 4.2], [3.5, 6], [4.8, 5.9], [6.4, 6.1], [7.8, 4.5], [10.4, 8.5], [10.8, 6.2]], \
              color = 'yellow', marker = '*', size = 90)
show(backdrop+p1+p2+p3+p4+pts, ymax = 10)
```



## Objectives for today:

1. Derivatives
2. Differential equations
3. Initial value problems

### 1.1 1. Derivatives

As you've been learning in lectures, completing derivatives by hand can be quite tricky. In Sage, it is mercifully easy! The command is `derivative()` or `diff()` (reminder that all the commands for the course are described in the *Glossary of Commands* on blackboard!), which takes two arguments: (1) the function you're differentiating, and (2) the variable you're differentiating with respect to.

```
[2]: y = 4^(x-ln(x))*arctan(x)/sin(x+e^x)
      show(y)
      show(diff(y, x)) # I would not want to try doing that by hand!
```

$$4^{(x - \log(x))} \arctan(x) / \sin(x + e^x)$$

$$-2 \cdot 4^{(x - \log(x))} \cdot (1/x - 1) \cdot \arctan(x) \cdot \log(2) / \sin(x + e^x) - 4^{(x - \log(x))} \cdot (e^x + 1) \cdot \arctan(x) \cdot \cos(x + e^x) / \sin(x + e^x)^2 + 4^{(x - \log(x))} / ((x^2 + 1) \sin(x + e^x))$$

There is an optional third argument in which you can indicate the number of times you're taking the derivative (the default is 1, of course). For example, if you needed the third derivative  $\frac{d^3y}{dx^3}$ , you would put the number 3 in the third argument.

Find the first and second derivatives of  $f(x) = \frac{\sin(x)}{4x}$ .

[ ]:

Any questions? Hopefully this part was pretty straight-forward.

## 1.2 2. Differential Equations

In differential equations; variables, functions, and the derivatives of these functions can be intertwined into an equation together. Linear, ordinary differential equations that can be solved with the command **desolve**.

We previously learned about the **solve** function, and **desolve** works in a very similar way. The first entry is for our equation, but the second entry is the dependent variable (often  $x$ , or whatever takes its place in the  $\frac{dy}{dx}$  notation). Our goal is to figure out what the function  $y$  is.

The first step is to set up your variables so that one is assigned as a **function** of the other. It looks like this:

```
[2]: # for some function f using variable x, we would write:
x = var('x')
f = function('f')(x)
show(f)
# now f is an arbitrary function of x.
```

$f(x)$

It will appear this way throughout your outputs, so don't be confused and think it's multiplying the function  $f$  by a variable  $x$ . It is just indicating that  $f$  is a *function of  $x$* .

A differential equation could look something like this:

```
[26]: df = diff(f, x)
show(df) # this is the derivative of an arbitrary function f with respect to x.
# Sage's presentation looks a little weird, but you can imagine the f(x)
# being in the numerator of the fraction instead of on the side.

show( (x+3)*df == e^f)
# don't forget the double == !
```

$\text{diff}(f(x), x)$

$(x + 3) * \text{diff}(f(x), x) == e^{f(x)}$

Notice how the derivative of  $f$ ,  $\frac{df}{dx}$ , is part of an equation that involves  $f$  and  $x$ .

Let's now set up the differential equation  $\frac{dy}{dx} = y$ . In other words, what function  $y$  is equal to its own derivative?

[ ]:

The `_C` part just means that you could put a coefficient in front and the equation would still hold. This is what we call a *general solution* to the differential equation.

One more example. What is the general solution of  $f$  for which  $\frac{df}{dx} = \frac{1}{3}f^{-2}$ ?

[ ]:

In this case, Sage leaves us with a tiny bit of work to do. We can isolate  $f(x)$  by taking the cube root on both sides.

Just watch out for instances where your function isn't completely isolated in your output!

### 1.2.1 Exercise questions

Using the approach from above, solve for the general solutions for the functions in the given differential equations.

1.  $\frac{dy}{dx} = x^2 + x$

2.  $2\left(\frac{dh}{dx}\right) = h + x$

[ ]:

[ ]:

## 1.3 3. Initial Value Problems

In the previous exercises we always get a constant  $C$  in our output.  $C$  can be any real number, which means there are an infinite number of versions of the function which satisfy the differential equation. Initial value problems require us to find a unique solution from that general group of solutions which satisfy initial conditions ("ics") that are provided in the question.

We will again use **desolve** with the added argument **ics** = [**a**, **b**] to get the specific solution. Here **a** is the initial condition of the independent variable, and **b** is the initial condition of the function. So if we had a differential equation using  $\frac{dy}{dx}$ , **a** would refer to  $x$  and **b** would refer to  $y$ .

Using the same  $\frac{df}{dx} = \frac{1}{3}f^{-2}$  as before, now solve for a unique solution where  $x = 1$  and  $f = -1$ .

[1]: `# copy and paste your work from above`

What if  $x = 2$  and  $f = 4$ ?

[ ]:

In summary, here are the steps to solving a differential equation in Sage: 1. Define your independent variable with `var(_)`, and the function of that variable with `function(_)`. 2. Define the derivative term (i.e.  $\frac{dy}{dx}$  or  $\frac{dg}{dx}$ , whatever it may be). 3. Recreate the differential equation given in the problem. 4. Put the equation in the first argument of `desolve(_)`, and the function in the second argument. 5. Identify initial conditions in the third argument, if asked for a particular solution. 6. Check your output to ensure your function has been completely solved for.

## 2 Exercise questions

3. Using the differential equation from Exercise question 1, find the unique solution that accompanies initial conditions  $x = -2$  and  $y = 1$ .
4. Using  $2(\frac{dh}{dx}) = h + x$  from Exercise question 2, find the unique solution for  $h$  where  $x = 0$  and  $h = 0$ .

[ ]:

[ ]:

There is **NO** lab next week. We'll see you here again after reading week :)

### 2.1 Reminders

Regular office hours generally are cancelled over reading week. If you'd like in-person (or zoom) support, email ahead with your prof or TA to confirm their availability.

Maya will be out of town next week (14 - 17 october), but will be checking email and holding a virtual office hour on wednesday the 15th if you need support. More details will be in a Blackboard announcement ... eventually.

Assignment 3 is due 17 october.

[ ]: