Today: 21/11/25

Continuing ex:3 from Tuesday

$$\boxed{\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}dx} = \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}(-1)du = \int_0^{\pi} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}du$$

$u = \pi - x$

$x = \pi - u$

$dx = (-1)du$

x	u
0	π
π	0

$$\sin(\pi-u) = \cancel{\underbrace{\sin(\pi)}_{=0}\cos(u)} - \sin(u)\underbrace{\cos(\pi)}_{=-1}$$
$$= \sin(u)$$
$$\cos(\pi-u) = \underbrace{\cos(\pi)}_{=-1}\cos(-u) - \cancel{\overbrace{\sin(\pi)}^{=0}\sin(-u)}$$
$$= -\cos(u)$$

$$= \int_0^{\pi} \frac{(\pi-u)\sin(u)}{1+(-\cos(u))^2}du = \int_0^{\pi} \frac{(\pi-u)\sin(u)}{1+\cos^2(u)}du$$

$$= \int_0^{\pi} \frac{\pi\sin(u)}{1+\cos^2(u)}du - \int_0^{\pi} \frac{u\sin(u)}{1+\cos^2(u)}du$$

$$= \int_0^{\pi} \frac{\pi\sin(u)}{1+\cos^2(u)}du - \boxed{\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}du}$$

$$2\left[\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}du\right] = \int_0^{\pi} \frac{\pi\sin(u)}{1+\cos^2(u)}du$$

$$\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}du = \frac{1}{2}\int_0^{\pi} \frac{\pi\sin(u)}{1+\cos^2(u)}du$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin(u)}{1+\cos^2(u)}du$$

$w = \cos(u)$

$dw = -\sin(u)du$

$(-1)dw = \sin(u)du$

u	w
0	1
π	-1

$$= \frac{\pi}{2}\int_1^{-1} \frac{-1}{1+w^2}dw$$

$$= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+w^2}dw$$

← switched to get the 1 on top as positive

★ put arctan on cheat sheet

$$= \frac{\pi}{2}\left[\arctan(w)\Big|_{-1}^{1}\right]$$

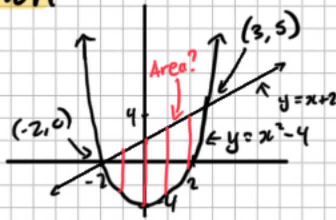
$$= \frac{\pi}{2}\left[\arctan(-1) - \arctan(1)\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi^2}{4}$$

Applications of integration

Find the area of the region below the line $y=x+2$ and above the parabola $y=x^2-4$



Area between $f(x)$ and $g(x)$ from $x=a$ to $x=b$

$$\int_a^b (\text{upper}-\text{lower})\,dx$$

$$x^2-4=x+2$$
$$(x+2)(x-2)=x+2$$
$$x-2=1$$
$$x=3$$

$$A=\int_{-2}^{3}\left((x+2)-(x^2-4)\right)dx$$

$$=\int_{-2}^{3}(-x^2+x+6)\,dx=\left(-\frac{x^3}{3}+\frac{x^2}{2}+6x\right)\Big|_{-2}^{3}$$

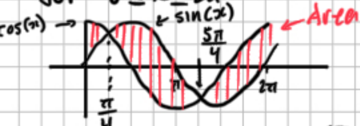
$$=\left(-\frac{3^3}{3}+\frac{3^2}{2}+6(3)\right)-\left(-\frac{(-2)^3}{3}+\frac{(-2)^2}{2}+6(-2)\right)$$

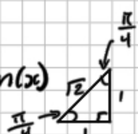
$$=\left(-9+\frac{9}{2}+18\right)-\left(\frac{8}{3}+\frac{4}{2}-12\right)$$

$$=\frac{27}{2}+\frac{22}{3}$$

$$=\frac{125}{6}$$

Find the area between $y=\cos(x)$ and $y=\sin(x)$ for $0\le x\le 2\pi$





$$A=\int_0^{2\pi}(\text{upper}-\text{lower})\,dx$$

$$=\int_0^{\pi/4}(\cos(x)-\sin(x))\,dx+\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(\sin(x)-\cos(x))\,dx+\int_{\frac{5\pi}{4}}^{2\pi}(\cos(x)-\sin(x))\,dx$$

$$=\underbrace{(\sin(x)+\cos(x))}\Big|_0^{\frac{\pi}{4}}+\underbrace{(-\cos(x)-\sin(x))}\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}+(\sin(x)+\cos(x))\Big|_{\frac{5\pi}{2}}^{2\pi}$$

$$=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-(0+1)+\left(-\frac{-1}{\sqrt{2}}-\frac{-1}{\sqrt{2}}\right)-\left(\frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)+(0+1)-\left(\frac{-1}{\sqrt{2}}+\frac{-1}{\sqrt{2}}\right)$$

$$=\frac{8}{\sqrt{2}}=4\sqrt{2}$$

Basic properties of antiderivatives

1° $\int c f(x)\,dx = c\int f(x)\,dx$

2° $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

3° (Integration by parts)

$$\int u\cdot v'\,dx = uv - \int u'\cdot v\,dx$$

$$\boxed{\begin{aligned}(uv)' &= u'v + uv'\\ u\cdot v' &= (uv)' - u'v\end{aligned}}$$

* goal = achieve something simpler

ex: $\int x e^{x}\,dx$

$u = x \quad v' = e^x$

$u' = 1 \quad v = e^x$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x + C$

4° (Basic substitution Rule)

$$\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + C$$

where $F'(t) = f(t)$

$u = g(x) \quad \frac{du}{dx} = g'(x)$

$du = g'(x)\,dx$

$= \int f(u)\,du$

$= F(u) + C = F(g(x)) + C$

$\int 2xe^{x}\,dx = \int e^{x}\,du = e^{u} + C$

$= e^{x^2} + C$

$u = x^2$ ← antiderivative

$du = 2x\,dx$

$u = x^4 - 3x^2 + 6 \quad du = (4x^3 - 6x)\,dx$

$\int (x^4 - 3x^2 + 6)(4x^3 - 6x)\,dx$

$\int u\,du = \frac{u^2}{2}$ ← substitute $x^4 - 3x^2 + 6$ back in

antiderivative

$$\boxed{\frac{u^{1+1}}{1+1} = \frac{u^2}{2}}$$