Today: 14/11/25

Integration - substitution and integration by parts

Q1:

$$\int_1^e \frac{\ln(x^2)}{x}\,dx \longrightarrow \int_0^2 u\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{u^2}{2}\Big|_0^2$$

$$= \frac{u^2}{4}\Big|_0^2 = \frac{2^2}{4} - \frac{0^2}{4} = 1$$

$u = \ln(x^2)$

$\frac{du}{dx} = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$

$du = \frac{2}{x}\,dx$

$\frac{1}{2}\,du = \frac{1}{x}\,dx$

x	u
1	0
e	2

↑ makes it simpler

Q2: $\int \frac{\cos(x) - \sin(x)}{1 + 2\sin(x)\cos(x)}\,dx$

$$= \int \frac{\cos(x) - \sin(x)}{\cos^2(x) + \sin^2(x) + 2\sin(x)\cos(x)}\,dx$$

$$= \int \frac{\cos(x) - \sin(x)}{(\cos(x) + \sin(x))^2}\,dx = \int \frac{1}{u^2}\,du$$

$u = \cos(x) + \sin(x)$

$\frac{du}{dx} = -\sin(x) + \cos(x)$

$du = (\cos(x) - \sin(x))\,dx$

$$= \frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{u^{-1}}{-1} + C$$

$$= \frac{-1}{u} + C$$

$$= \frac{-1}{\cos(x) + \sin(x)} + C$$

Q3: $\int \frac{6x+1}{\sqrt{3x^2+x+1}}\,dx$

$w = 3x^2 + x + 1$ ← making it the substitute because it's $\frac{d}{dx}$ is the numerator

$dw = (6x+1)\,dx$

$$= \int \frac{1}{\sqrt{w}}\,dx = \int w^{-\frac{1}{2}}\,dx = \frac{w^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{w^{\frac{1}{2}}}{1/2} + C$$

$$= 2\sqrt{w} + C$$

$$= 2\sqrt{3x^2 + x + 1} + C$$

Q4: $\int \frac{2e^x}{\sqrt{e^{2x}+9}}\,dx = \int \frac{2}{\sqrt{u^2+9}}\,du = \int \frac{6\sec^2(\theta)}{\sqrt{(3\tan(\theta))^2+9}}\,d\theta$

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$u = 3\tan(\theta)$

$du = 3\sec^2(\theta)\,d\theta$

$$= \int \frac{6\sec^2(\theta)}{\sqrt{9\tan^2(\theta)+9}}\,d\theta$$

$$= \int \frac{6\sec^2\theta}{\sqrt{9(\tan^2\theta+1)}}\,d\theta$$

$$= \int \frac{6\sec^2\theta}{3\sqrt{\sec^2\theta}}\,d\theta$$

$$= \int \frac{2\sec^2(\theta)}{\sec(\theta)}\,d\theta$$

$$= 2\int \sec(\theta)\,d\theta$$

$$= 2\ln(\sec(\theta) + \tan(\theta)) + C$$

$$= 2\ln\left(\sqrt{\left(\frac{u}{3}\right)^2 + 1} + \frac{u}{3}\right) + C$$

$$= 2\ln\left(\sqrt{\left(\frac{e^x}{3}\right)^2 + 1} + \frac{e^x}{3}\right) + C$$

Product rule for derivatives

$$(u\cdot v)' = u'\cdot v + u\cdot v'$$

$$\Rightarrow u\cdot v' = (u\cdot v)' - u'\cdot v$$

$$\Rightarrow \int u\cdot v'\,dx = \int (u\cdot v)'\,dx - \int u'\cdot v\,dx$$

$$\boxed{\int u\cdot v'\,dx = u\cdot v - \int u'\cdot v\,dx}$$

ex 1: $\int x\cos(x)\,dx = x\sin(x) - \int 1\cdot \sin(x)\,dx$

$u = x \quad v' = \cos(x)$ $\qquad = x\sin(x) - [-\cos(x)] + C$

$u' = 1 \quad v = \sin(x)$ $\qquad = x\sin(x) + \cos(x) + C$

$$\frac{d}{dx}\left(x\sin(x) + \cos(x) + C\right)$$

$$= 1\cdot \cancel{\sin(x)} + x\cos(x) + (\cancel{-\sin(x)}) + 0$$

$$= x\cos(x)$$

ex 2: $\int x^2\ln(x)\,dx = \frac{x^3}{3}\ln(x) - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$

$u = \ln(x) \quad v' = x^2$ $\qquad = \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx$

$u' = \frac{1}{x} \quad v = \frac{x^3}{3}$ $\qquad = \frac{x^3}{3}\ln(x) - \frac{1}{3}\cdot\frac{x^3}{3} + C$

$$= \frac{x^3}{3}\left(\ln(x) - \frac{1}{3}\right) + C$$

ex 3: $\int_0^1 x^2e^x\,dx = x^2e^x\Big|_0^1 - \int_0^1 2xe^x\,dx$ ← integrate by parts again

$u = x^2 \quad v' = e^x$ $\qquad s = 2x \quad t' = e^x$

$u' = 2x \quad v = e^x$ $\qquad s' = 2 \quad t = e^x$

$$= 1^2e^1 - \cancel{0^2e^0} - \left[2xe^x\Big|_0^1 - \int_0^1 2e^x\,dx\right]$$

$$= e - \left[(2\cdot 1\cdot e^1 - 2\cdot 0\cdot e^0) - 2e^x\Big|_0^1\right]$$

$$= e - \left[2e - (2e^1 - 2e^0)\right]$$

$$= e - [2] = \underline{e-2}$$