Today: 11/11/25

Integration - Basic substition rule

chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

substitution: $\int f'(g(x)) \cdot g'(x)\, dx$

$$\int f'(u)\, du = f(u) + C$$
$$f(g(x)) + C$$

ex 1: $\int x e^{x^2} dx = \int e^u \cdot \frac{1}{2} du = \frac{1}{2} e^u + C$

$= \frac{1}{2} e^{x^2} + C$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

substitute to get that

$$\int_0^1 x\sqrt{1+x^2}\, dx = \int_{x=0}^{x=1} \sqrt{1+w^2} \cdot \frac{1}{2} dw$$

$\boxed{w = x^2 \quad dw = 2x\,dx \quad \frac{1}{2} dw = x\,dx}$

$$= \frac{1}{2} \int_{x=0}^{x=1} \sqrt{1+w} \cdot dw$$

$u = 1 + w \quad \boxed{du = dw}$

$$= \frac{1}{2} \int_{x=0}^{x=1} u^{\frac{1}{2}} du$$

$$= \frac{1}{\cancel{2}} \cdot \frac{u^{3/2}}{3/\cancel{2}}$$

$$= \frac{1}{3}(1+w)^{\frac{3}{2}} \Big|_{x=0}^{x=1}$$

$$= \frac{1}{3}(1+x^2)^{3/2} \Big|_0^1$$

$$= \frac{1}{3}(1+1^2)^{3/2} - \frac{1}{3}(1+0^2)^{\frac{3}{2}}$$

$$= \frac{2\sqrt{2}}{3} - \frac{1}{3}$$

* faster method

$$\int_0^1 x\sqrt{1+x^2} \cdot dx = \frac{1}{2}\int_1^2 \sqrt{v} \cdot dv = \frac{1}{2}\int_1^2 v^{\frac{1}{2}} dv$$

$v = 1 + x^2$

$dv = 2x\,dx$

$\frac{1}{2} dv = x\,dx$

x	v
0	1
1	2

$$= \frac{1}{\cancel{2}} \cdot \frac{u^{3/2}}{3/\cancel{2}} \Big|_1^2$$

$$= \frac{2^{3/2}}{3} - \frac{1^{3/2}}{3}$$

$$= \frac{2\sqrt{2}}{3} - \frac{1}{3}$$

Ex 2: $\int_0^{\ln(2)} \frac{e^x}{e^{2x}+1} dx = \int_1^2 \frac{1}{u^2+1} dx = \arctan(u) \Big|_1^2$

$= \arctan(2) - \arctan(1)$

$u = e^{2x}$

$du = e^{2x} dx$

$e^{2x} + 1 = (e^x)^2 + 1$

$= u^2 + 1$

Ex 3: $\int \frac{1}{\sqrt{x^2+1}} dx = \int \frac{1}{\sqrt{\tan^2\theta+1}} \cdot \sec^2(\theta)\, d\theta$

$x = \tan(\theta)$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$\frac{dx}{d\theta} = \frac{d}{dx}(\tan\theta)$

$= \sec^2(\theta)$

$= \int \frac{1}{\sqrt{\sec^2\theta}} \sec^2\theta\, d\theta$

$= \int \sec\theta\, d\theta$

$= \ln(\sec(\theta) + \tan(\theta)) + C$

$= \ln(\sqrt{x^2+1} + x) + C$