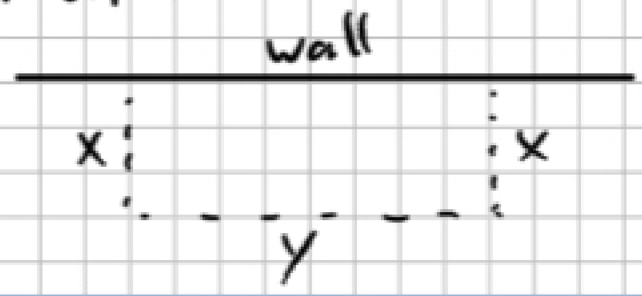
Today: 31/10/25

Optimization & related rates

Ex:



rectangular plot
100m fencing
What is the max area?

Area $= xy$ Fence: $y + 2x = 100$

$y = 100 - 2x$

$A = x(100-2x)$

$= 100x - 2x^2 \longrightarrow \frac{dA}{dx} = \frac{d}{dx}(100x - 2x^2)$

$= 100 - 4x$

$0 = 100 - 4x$

$4x = 100$

$x = 25$

$A(25) = 100(25) - 2(25)^2$

$= 2500 - 1250$

$= 1250m^2$

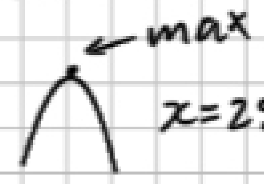
checking for max or min

$\hookrightarrow 100 - 4x > 0$ ↑

$\Leftrightarrow x < 25$

$100 - 4x < 0$

$\Leftrightarrow x > 25$ ↓



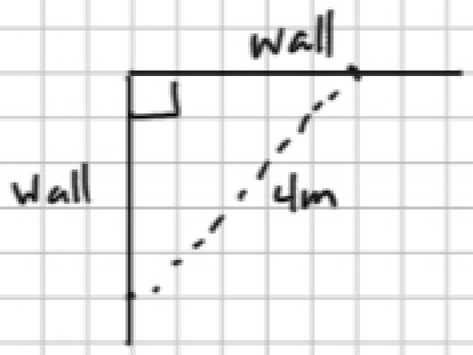
$x = 25$

domain: $50 > x > 0 \rightarrow (0, 50)$

check only for something funky

at the endpoints $\rightarrow \lim_{x \to 0} (100x - 2x^2) = 0$

$\lim_{x \to 50} (100x - 2x^2) = 100(50) - 2(50)^2 = 0$



Q1: What is the max area of the triangular plot if the fence is 4m?

Q2: What is the least length of fence to get to an area of $27m^2$?

Q1: Area: $\frac{b \cdot h}{2}$ $\quad b^2 + h^2 = 16$

$b = \sqrt{16 - h^2}$

$= \frac{h}{2}\sqrt{16-h^2}$

$= \frac{d}{dx}\left(\frac{h}{2}\right)\left(\sqrt{16-h^2}\right) + \frac{d}{dx}\left(\sqrt{16-h^2}\right)\left(\frac{h}{2}\right)$

$= \frac{1}{2}\sqrt{16-h^2} + \left(\frac{1}{2}(16-h^2)^{-\frac{1}{2}}(-2h)\right)\left(\frac{h}{2}\right)$

$= \frac{1}{2}\sqrt{16-h^2} + \frac{h}{4} \cdot \frac{1}{\sqrt{16-h^2}} \cdot (-2h)$

$= \frac{1}{2}\underbrace{\left(\sqrt{16-h^2} - \frac{h^2}{\sqrt{16-h^2}}\right)}_{0}$

$\frac{\sqrt{16-h^2}}{\sqrt{16-h^2}} \cdot \frac{\sqrt{16-h^2}}{1} - \frac{h^2}{\sqrt{16-h^2}} = 0$

$\frac{16-h^2}{\sqrt{16-h^2}} - \frac{h^2}{\sqrt{16-h^2}}$

$\frac{16-h^2-h^2}{\sqrt{16-h^2}} = 0$

$16 - h^2 - h^2 = 0$

$16 - 2h^2 = 0$

$\Leftrightarrow (16 - h^2) - h^2 = 0$

$\Leftrightarrow 16 - 2h^2 = 0$

$16 = 2h^2$

$8 = h^2$

only concerned with when positive $\rightarrow \sqrt{8} = h$

$2\sqrt{2} = h$

$A(2\sqrt{2}) = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}\sqrt{16 - (2\sqrt{2})^2}$

$= \sqrt{2}\sqrt{16-8}$

$= \sqrt{2}\sqrt{8}$

$= \sqrt{2} \cdot 2\sqrt{2}$

$= 4$

$0 < h < 4$

$h \to 0 \quad A(h) \to 0$

$h \to 4 \quad A(h) \to 4$

$\therefore$ the max is at $x = 2\sqrt{2}$ & $A(2\sqrt{2}) = 4m^2$

Q2. $A = \frac{bh}{2} = 27 \quad b^2 + h^2 = x^2$

& solving for x

$b = \frac{54}{h}$

$\hookrightarrow \left(\frac{54}{h}\right)^2 + h^2 = x^2$ $\quad$ minimize x

only concerned with positive $\rightarrow +\sqrt{\left(\frac{54}{h}\right)^2 + h^2} = x$

$\left[\left(\frac{54}{h}\right)^2 + h^2\right]^{\frac{1}{2}} = x$

$\frac{d}{dx} 54^2 \cdot h^{-2}$

$= -2 \cdot 54^2 \cdot h^{-3}$

$= \frac{-2 \cdot 54^2}{h^3}$

$\frac{dx}{dh} = \left(\frac{1}{2}\left(\left(\frac{54}{h}\right)^2 + h^2\right)^{-\frac{1}{2}}\right) \cdot \frac{d}{dh}\left(\frac{54^2}{h^2} + h^2\right)$

$= \frac{1}{\cancel{2}\sqrt{\left(\frac{54}{h}\right)^2 + h^2}} \cdot \left(-\cancel{2}\frac{54^2}{h^3} + \cancel{2}h\right)$

$0 = \frac{-\frac{54^2}{h^3} + h}{\sqrt{\left(\frac{54}{h}\right)^2 + h^2}}$

$\Leftrightarrow -\frac{54^2}{h^3} + h = 0$

$\Leftrightarrow \frac{-54^2 + h^4}{h^3} = 0$

$\Leftrightarrow -54^2 + h^4 = 0$

$\Leftrightarrow h^{\cancel{4}2} = 54^{\cancel{2}}$

$h = \sqrt{54}$

$x(54) = \sqrt{\frac{54^2}{(\sqrt{54})^2} + (\sqrt{54})^2}$

$= \sqrt{2 \cdot 54}$

$= \sqrt{108}$

Max or min? (or neither)

$0 < h < \infty$

$h \to 0 \quad x \to \infty$

$h \to \infty \quad x \to \infty$

$\therefore x(\sqrt{54}) = \sqrt{108}$ is a minimum