Today: 14/10/25

"Qualitative Analysis" - "curve sketching"

First up, $F(x) = \frac{x}{1-x^2}$, what can we find out about the function?

0° Domain → all x, except $x = 1$

$\mathbb{R} \in \{-1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

1° Intercepts → y int: $y = f(0) = 0$

x int: $y = f(x) = 0 = \frac{x}{1-x^2} \iff x = 0$

2° Horizontal asymptotes

$$\lim_{x \to -\infty} \frac{x}{1-x^2} = \lim_{x \to -\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx}(1-x^2)} = \lim_{x \to -\infty} \frac{1 \to 1}{-2x \to \infty} = 0^+$$

approach from above

$$\lim_{x \to \infty} \frac{x}{1-x^2} = \ldots = \lim_{x \to \infty} \frac{1 \to 1}{-2x \to -\infty} = 0^-$$

approach from negative side

3° vertical asymptotes

- look at points where $f(x)$ is undefined
  - i.e. $x = 1$ and $x = -1$

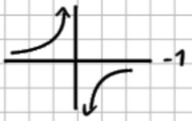
$$\lim_{x \to -1^-} \frac{x}{1-x^2} = \frac{x \to -1}{1-x^2 \to 0^-} = \infty$$

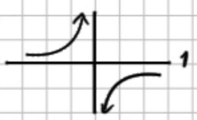
from the left

$$\lim_{x \to -1^+} \frac{x}{1-x^2} = \frac{x \to -1}{1-x^2 \to 0^+} = -\infty$$

$$\lim_{x \to 1^-} \frac{x}{1-x^2} = \frac{x \to 1}{1-x^2 \to 0^+} = \infty$$

$$\lim_{x \to 1^+} \frac{x}{1-x^2} = \frac{x \to 1}{1-x^2 \to 0^-} = -\infty$$





4° Max/Min/Inc./dec.

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1-x^2}\right)$$

$$= \frac{(x)'(1-x^2) - (x)(1-x^2)'}{(1-x^2)^2}$$

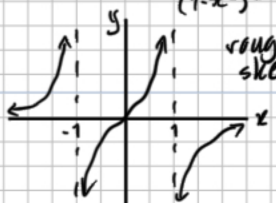
$$= \frac{1-x^2-(x)(-2x)}{(1-x^2)^2}$$

$$= \frac{1+x^2}{(1-x^2)^2}$$

↑ = increasing

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$f'(x)$	+	$\wr$	+	$\wr$	+
$f(x)$	↑	$\wr$	↑	$\wr$	↑

undefined



rough sketch

∴ no max or min

↳ only intervals of inc.

5° concavity and inflection

$$f''(x) = \frac{d^2}{dx^2}\left(\frac{x}{1-x^2}\right) = \frac{d}{dx}\left(\frac{1+x^2}{(1-x^2)^2}\right)$$

$$= \frac{(1+x^2)'(1-x^2)^2 - (1+x^2)((1-x^2)^2)'}{((1-x^2)^2)^2}$$

$$= \frac{2x(1-x^2)^2 - (1+x^2)(2(1-x^2)(-2x))}{((1-x^2)^2)^2}$$

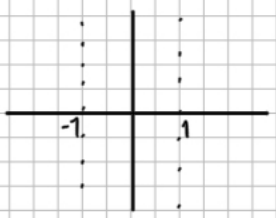
$$= \frac{\cancel{(1-x^2)}(2x(1-x^2)) - (1+x^2)(-4x)}{((1-x^2)^2)^2}$$

$$= \frac{2x(1-x^2) - (1+x^2)(-4x)}{(1-x^2)^3}$$

$$= \frac{2x - 2x^3 + 4x + 4x^3}{(1-x^2)^3}$$

$$= \frac{2x^3 + 6x}{(1-x^2)^3}$$

$$= \frac{2x(3+x^2)}{(1-x^2)^3} = 0 \text{ when } x = 0$$



$$f''(x) = \frac{2x(x^2+3)}{(1-x^2)^3}$$

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$f''(x)$	+	$\wr$	−	0	+	$\wr$	−
$f(x)$	∪	$\wr$	∩		∪	$\wr$	∩

concave up

undefined

inflection point

concave down