Today: 10/10/25

Max/Min

Q. How do we find a local max or min of $y = f(x)$ on some interval?

1. look for the points where $f'(x) = 0$ or is undefined

2. look at how $f'(x)$ behaves near these "critical points" [First derivative]

If $f'(x) < 0$ then the point, of $f'(x) > 0$ after the point... then it's a local min

If $f'(x) > 0$ then the point of $f'(x) < 0$ after the point... then it's a local max

2'. (other option) look at $f''(x) = \_\_\_\_$ at the point in question. $\frac{d}{dx} f'(x)$

$$\boxed{\begin{array}{l} f''(x) = 0 \Rightarrow \text{no info} \\ f''(x) > 0 \text{ local min} \\ f''(x) < 0 \text{ local max} \end{array}}$$

Ex: $y = 3x^3 + 2x$ on -5 to 5

find the local max/min of y on the interval

$\frac{dy}{dx} = 9x^2 + 2 \longrightarrow \pm\sqrt{-\frac{2}{9}} = x$ $\therefore$ no solution... no critical points ↳ so no max or min

solve for 0

x	$(-\infty, \infty)$
$\frac{dy}{dx}$	+
y	↑

at the endpoints $(-5, 5)$

$3 \cdot 5^3 + 2 \cdot 5 = 385$

$3 \cdot (-5)^3 + 2 \cdot (-5) = -385$

↳ the max of y on $[-5, 5]$ is 385 (at $x = 5$) and min is -385 (at $x = -5$)

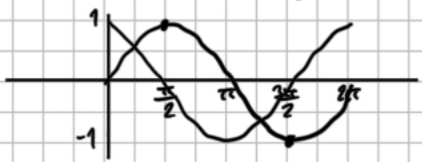
Ex: $y = \sin(x)$ on 0 to 2π

find the local max/min of y on the interval

$\frac{dy}{dx} = \cos(x)$

$= 0$

when $x = \frac{\pi}{2}$ & $\frac{3\pi}{2}$



2nd derivative test

$\frac{dy^2}{dx^2} = \frac{d}{dx} \cos(x) = -\sin(x)$

$-\sin\left(\frac{\pi}{2}\right) = -1 \Rightarrow$ local max

$-\sin\left(\frac{3\pi}{2}\right) = 1 \Rightarrow$ local min

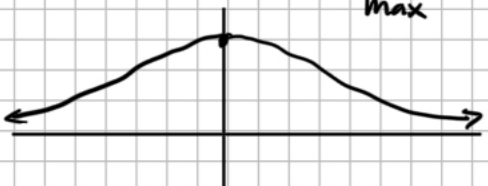
Ex: $y = 2^{-x^2}$ on $-\infty$ to ∞

$\frac{dy}{dx} = \ln(2) \cdot 2^{-x^2} \cdot \frac{d}{dx}(-x^2)$

$= \ln(2) \cdot 2^{-x^2} \cdot (-2x)$

$= 0$ when $x = 0 \longrightarrow$ > 0 when $x < 0$, < 0 when $x > 0$

x	$(-\infty, \infty)$	0	$(0, \infty)$
y'	+	0	−
y	↑	max	↓



Q. Does this have a lower bound.

↳ we know it never achieves a min.

$\lim_{x \to -\infty} 2^{-x^2} = \lim_{x \to -\infty} \frac{1}{2^{x^2}} = 0^+$

$\lim_{x \to \infty} 2^{-x^2} = \lim_{x \to \infty} \frac{1}{2^{x^2}} = 0^+$

What can we find out (about the graph) of $f(x) = \frac{x}{1+x^2}$

1° Domain is $(-\infty, \infty)$, because the denominator is never $= 0$

($f(x)$ is also continuous and differentiable for all x)

2° intercepts: y int: $f(0) = \frac{0}{1+0^2} = 0$

x int: $0 = \frac{x}{1+x^2}$: $x = 0$

3° Horizontal asymptote(s)?

$$\lim_{x\to\infty} \frac{x}{1+x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(1+x^2)} = \lim_{x\to\infty} \frac{1 \to 1}{2x \to \infty} = 0^+$$

approaching 0 from above

$$\lim_{x\to-\infty} \frac{x}{1+x^2} = \lim_{x\to-\infty} \overset{\text{same}}{- \text{''} -} = \lim_{x\to-\infty} \frac{1 \to 1}{2x \to -\infty} = 0^-$$

approaching 0 from below

4° Vertical asymptote(s)?

none: it is defined and acts for all x

5° Max/Min/inc./dec.

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1+x^2}\right)$$

$$= \frac{(x)'(1+x^2) - (x)(1+x^2)'}{(1+x^2)^2}$$

$$= \frac{1 \cdot 1 + x^2 - x \cdot 2x}{(1+x^2)^2}$$

$$= \frac{1 + x^2 - 2x^2}{(1+x^2)^2}$$

$$= \frac{1 - x^2}{(1+x^2)^2}$$

$= 0 \iff x \pm 1$

$> 0 \iff 1 - x^2 > 0$

$1 > x^2$

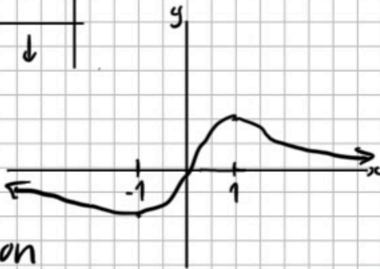
$-1 < x < 1$

$< 0 \iff 1 - x^2 < 0$

$x^2 < 0$

$x > 1$ or $x < -1$

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\downarrow$	min	$\uparrow$	max	$\downarrow$



6. Concavity and inflection

concave down

concave up

$$f''(x) = \frac{d}{dx}\left(\frac{1-x^2}{(1+x^2)^2}\right)$$

$$= \frac{(1-x^2)'(1+x^2)^2 - (1-x^2)((1+x^2)^2)'}{((1+x^2)^2)^2}$$

$$= \frac{-2x\cdot(1+x^2)^2 - (1-x^2)(2(1+x^2)(2x))}{(1+x^2)^4}$$

$$= \frac{\cancel{(1+x^2)}(-2x(1+x^2)) - (1-x^2)(4x)}{(1+x^2)^{\cancel{4}\,3}}$$

$$= \frac{-2x-2x^3-4x+4x^3}{(1+x^2)^3}$$

$$= \frac{-6x+2x^3}{(1+x^2)^3}$$

$$= \frac{2x(x^2-3)}{(1+x^2)^3}$$

$$= \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3}$$

$=0 \iff x=0$ or $x=\sqrt{3}$ or $x=-\sqrt{3}$

$>0 \iff$ $<-\sqrt{3}$ $\quad -\sqrt{3}<0$

$-$ $\quad +$

$0<\sqrt{3}$ $\quad \sqrt{3}<$

$-$ $\quad +$

$<0 \iff$

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\cap$	inf	$\cup$	inf	$\cap$	inf	$\cup$