Today: 03/10/25

Continued derivatives

Ex1: $\frac{d}{dx} \ln\left(\frac{x^2+3x+17}{x-13}\right)$

$$= \frac{1}{\frac{x^2+3x+17}{x-13}} \cdot \frac{d}{dx}\left(\frac{x^2+3x+17}{x-13}\right)$$

$$= \frac{x-13}{x^2+3x+17} \cdot \frac{(x^2+3x+17)'(x-13)-(x^2+3x+17)(x-13)'}{(x-13)^2}$$

$$= \frac{\cancel{x-13}}{x^2+3x+17} \cdot \frac{(2x+3)(x-13)-(x^2+3x+17)}{(x-13)^{\cancel{2}\,1}}$$

$$= \frac{1}{x^2+3x+17} \cdot \frac{(2x^2-23x-39)-(x^2+3x+17)}{x-13}$$

$$= \frac{x^2-26x-56}{(x^2+3x+17)(x-13)}$$

another method:

$$\boxed{\begin{aligned}\ln\left(\tfrac{a}{b}\right) &= \ln(a)-\ln(b)\\ \ln(ab) &= \ln(a)+\ln(b)\end{aligned}}$$

$$\frac{d}{dx} \ln\left(\frac{x^2+3x+17}{x-13}\right)$$

$$= \frac{d}{dx}\left[\ln(x^2+3x+17) - \ln(x-13)\right]$$

$$= \frac{1}{x^2+3x+17} \cdot (x^2+3x+17)' - \frac{1}{x-13} \cdot (x-13)'$$

$$= \frac{2x+3}{x^2+3x+17} - \frac{1}{x-13}$$

$$= \frac{(2x+3)(x-13) - x^2+3x+17}{(x^2+3x+17)(x-13)}$$

$$= \frac{x^2-26x-56}{(x^2+3x+17)(x-13)}$$

Ex2: $\frac{d}{dx}\left((\sin(x)+\cos(x))^2 - (\sin(x)-\cos(x))^2\right)$

$$= \frac{d}{dx}\left((\cancel{\sin^2(x)}+2\sin(x)\cos(x)+\cancel{\cos^2(x)}) - (\cancel{\sin^2(x)}-2\sin(x)\cos(x)+\cancel{\cos^2(x)})\right)$$

$$= \frac{d}{dx}\left(4\sin(x)\cos(x)\right)$$

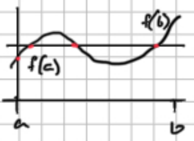
$$= \frac{d}{dx}\, 2\sin(2x)$$

$$= 2\cos(2x)\cdot 2$$

$$= 4\cos(2x)$$

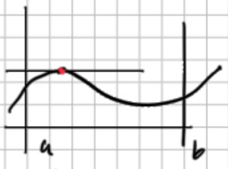
Intermediate value theorem

↳ if $f(x)$ is continuous on $[a,b]$ and d is a value between $f(a)$ and $f(b)$, then there is a $c \in [a,b]$ such that $f(c) = d$



Mean value theorem

↳ $f(x)$ is continuous on $[a,b]$ & differentiable on (a,b), then there is a c with $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$



At a local maximum or minimum a differentiable function has a derivative of 0

Find the local maximum and minimum of $y = x^3 - x$

$$\frac{d}{dx}(x^3-x) = 3x^2-1$$

$$= 0 \iff 3x^2-1$$

$$\iff x^2 = \frac{1}{3}$$

$$\iff x = \pm\frac{1}{\sqrt{3}}$$

how do we tell if these are maxima, or minimum, or neither?

try pluggin back into equation

$\hookrightarrow y = \left(\frac{-1}{\sqrt{3}}\right)^3 - \left(\frac{-1}{\sqrt{3}}\right)$

negative input = positive output

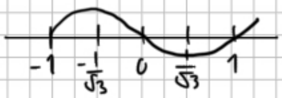
$= \frac{-1}{3\sqrt{3}} + \frac{1}{\sqrt{3}}$

$= \frac{2}{3\sqrt{3}}$

$y = \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right)$

$= -\frac{2}{3\sqrt{3}}$

positive input = negative output



$x = 0 \quad y = x^3 - x = 0$

$x = -1 \quad y = 0$

$x = 1 \quad y = 0$

local max $\Leftrightarrow \frac{d}{dx} > 0$ before

$\frac{d}{dx} < 0$ after

local min $\Leftrightarrow \frac{dy}{dx} < 0$ before

$\frac{dy}{dx} > 0$ after

Ex 3: $y = xe^{-x^2}$

$\frac{dy}{dx} = \frac{d}{dx}\left(x \cdot e^{-x^2}\right)$

$= (x)'\left(e^{-x^2}\right) + (x)\left(e^{-x^2}\right)'$

$= e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x)$

$= e^{-x^2} - 2x^2 \cdot e^{-x^2}$

$= e^{-x^2}(1 - 2x^2)$

$\frac{dy}{dx} = (1 - 2x^2)e^{-x^2}$ $\begin{matrix} >0 \\ =0 \\ <0 \end{matrix}$

$\Leftrightarrow 1 - 2x^2$ $\begin{matrix} >0 \\ =0 \\ <0 \end{matrix}$

make it equal to 0

$\frac{1}{2} = x^2$

$\Leftrightarrow$ $x^2 < \frac{1}{2}$

$x^2 = \frac{1}{2}$

$x^2 > \frac{1}{2}$

$\therefore f'(x)$ is positive when $x^2 < \frac{1}{2}$, i.e. $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

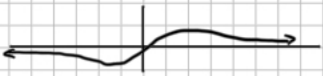
$\therefore f'(x)$ is negative when $x^2 > \frac{1}{2}$, i.e. $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$

$\therefore f'(x)$ equals 0 when $x^2 = \frac{1}{2} = \pm\frac{1}{\sqrt{2}}$

x	$(-\infty, -\frac{1}{\sqrt{2}})$	$-\frac{1}{\sqrt{2}}$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$(\frac{1}{\sqrt{2}}, \infty$
$f'(x)$	−	0	+	0	−
$f(x)$	↓	min	↑	max	↓

↓ = going down

↑ = going up



← graph of $y = xe^{-x^2}$

Ex 4: Find all the (local) maxima and/or minimum a of $y = x^3$

$\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2$

$y' = 3x^2$ $\begin{matrix} >0 \\ =0 \\ <0 \end{matrix}$ $\Leftrightarrow$ $\begin{matrix} x \neq 0 \\ x = 0 \\ \longrightarrow \text{never happens} \end{matrix}$

x	$(-\infty, 0)$	0	$(0, \infty)$
y'	+	0	+
y	↑	neither	↑