Today: 26/09/25

Basic derivatives / practice

0° Power Rule: $\frac{d}{dx} x^n = n x^{n-1}$

1° Trig identities: $\frac{d}{dx} \sin(x) = \cos(x)$

$\rightarrow \frac{d}{dx} \cos(x) = -\sin(x)$

$\rightarrow \frac{d}{dx} \tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$ ← quotient Rule

$$= \frac{\left[\frac{d}{dx}\sin(x)\right]\cos(x) - \sin(x)\left[\frac{d}{dx}\cos(x)\right]}{\cos^2(x)}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

$\rightarrow \frac{d}{dx} \sec(x) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$ ← quotient Rule

$$= \frac{\left[\frac{d}{dx}1\right]\cos(x) - 1\cdot\left[\frac{d}{dx}\cos(x)\right]}{\cos^2(x)}$$

$$= \frac{0\cdot\cos(x) - 1\cdot(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}$$

$$= \sec(x)\tan(x)$$

Explanation: $\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h}\right]$$

$$= \left[\lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h}\right] + \left[\lim_{h\to 0} \cos(x)\frac{\sin(h)}{h}\right]$$

$$= \sin(x)\underbrace{\left[\lim_{h\to 0}\frac{\cos(h)-1}{h}\right]}_{\text{special limit that equals 0}} + \cos(x)\underbrace{\left[\lim_{h\to 0}\frac{\sin(h)}{h}\right]}_{\text{special limit that equals 1}}$$

$$= \boxed{\cos(x)}$$

2° $\frac{d}{dx} e^x = e^x$ ← defined for all x

$\frac{d}{dx} \ln(x) = \frac{1}{x}$ $(x>0)$ ← defined for all positive x

proof: $\frac{d}{dx} e^x = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$

$= \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$ ← pull e^x out by factoring

$= e^x \lim_{h\to 0} \frac{e^h - 1}{h}$ ← special limit that equals 1

$= e^x \cdot 1$

$= e^x$

proof: $\frac{d}{dx} a^x = \ln(a) \cdot a^x$

$\frac{d}{dx} a^x = \frac{d}{dx} \left(e^{\ln(a)}\right)^x$

$= \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

where $f(t) = e^t$

$g(x) = \ln(a) \cdot x$

$= \frac{d}{dx}\left(e^{\ln(a)\cdot x}\right)$

$= e^{\ln(a)\cdot x} \cdot \frac{d}{dx}[\ln(a)\cdot x]$

$= a^x \cdot \ln(a) \cdot \underbrace{\frac{d}{dx} x}_{=1}$

$= a^x \cdot \ln(a)$

proof: $\frac{d}{dx} \log_a(x) = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(a)}\right] = \frac{1}{x} \cdot \frac{1}{\ln(a)} = \frac{1}{x \cdot \ln(a)}$

Other functions

3° Inverse trig functions ← Know for final exams

$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

$\arctan(\tan(x)) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}$

$\arccos(\cos(x)) = x$ if $0 \le x \le \pi$

$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

$\arcsin(\sin(x)) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

4° Hyperbolic & Inverse Hyperbolic functions

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$

derivative:

$\frac{d}{dx} \cosh(x) = \frac{1}{2}\left(\frac{d}{dx} e^x + \frac{d}{dx} e^{-x}\right)$

$= \frac{1}{2}\left(e^x + e^{-x} \cdot \frac{d}{dx}(-x)\right)$

$= \frac{1}{2}\left(e^x + e^{-x} \cdot (-1)\right)$

$= \frac{e^x - e^{-x}}{2}$ ← $\sinh(x)$