ε-δ and Computing Limits — Sept 1

For any $\varepsilon > 0$ there is a $\delta > 0$, such that if $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$

$\lim_{x\to a} f(x) = L$ The ε-δ game at a and L for f(x) is:

- two players A and B
- A moves first, then B, then A
  - A plays an $\varepsilon > 0$
  - B chooses a $\delta > 0$
  - A plays an x with $|x-a| < \delta$
- winning conditions:
  - A wins if $|f(x) - L| \geq \varepsilon$
  - B wins if $|f(x) - L| < \varepsilon$
- $\lim_{x\to a} f(x) = L$ means that B has a winning strategy
- $\lim_{x\to a} f(x) \neq L$ means that A has a winning strategy

$\lim_{x\to 2}(3x-5) = 11$ A: $\varepsilon = 1$ B: $\delta = \frac{1}{10}$ A: $e-0.7$

$|3\cdot(e-0.7)+5-11|$

$= |3\cdot 2.082\ldots - 6|$

$= |6.24 - 6|$

$= 0.24 < \varepsilon$ $\therefore$ B wins

Is there a winning strategy for B?

if A chooses $\varepsilon > 0$ we can backtrack

iff $|3x-6| < \varepsilon$

iff $3|x-2| < \varepsilon$

iff $|x-2| < \frac{\varepsilon}{3}$

B chooses $\delta = \frac{\varepsilon}{3}$ and A chooses x with $|x-2| < \frac{\varepsilon}{3}$

Computing Limits:

A function f(x) is continuous at a if $\lim_{x\to a} f(x) = f(a)$

Most functions are continuous, except maybe at some points

There are 3 types of discontinuity: point, line, and jump

The Practical Rules

9. If f(x) is continuous at a, $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2+2x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+1)} = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$

10. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

provided all these limits exist

2. $\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$

3. $\lim_{x\to a} cf(x) = c(\lim_{x\to a} f(x))$

4. $\lim_{x\to a} f(g(x)) = \lim_{t\to b} f(t)$ where $b = \lim_{x\to a} g(x)$

$\lim_{x\to 2} \sin(\sqrt{x^2+7}) = \lim_{t\to\sqrt{11}} \sin(t)$

$\lim_{x\to 2} \sqrt{x^2+7} = \sqrt{11} \quad = \sin\sqrt{11}$

Limits Practice

$$\lim_{x\to\pi} \frac{\sin(2x)}{\sin(x)} = \lim_{x\to\pi} \frac{2\sin(x)\cos(x)}{\sin(x)}$$
$$= 2\cos(\pi)$$
$$= 2(-1)$$
$$= -2$$

$$\lim_{x\to 0} e^{(1-\cos^2(x))} = e^{\lim_{x\to 0}(1-\cos^2 x)} \quad \text{or} \quad e^{\lim \sin^2 x} = e^{\sin^2(0)}$$
$$= e^{1-1^2} \qquad = e^{0^2}$$
$$= e^0 \qquad = e^0$$
$$= 1 \qquad = 1$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(1-\cos(x))}{\frac{d}{dx}(x)} \leftarrow \text{l'Hôpital's Rule (don't use)}$$
$$= \lim_{x\to 0} \frac{0-(-\sin x)}{1}$$
$$= \sin(0)$$
$$= 0$$

instead $\to$
$$= \lim_{x\to 0} \frac{1-\cos(x)}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)}$$
$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos x)}$$
$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos(x))}$$
$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$
$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right) \leftarrow = 0$$
$$= \lim_{x\to 0} \frac{\sin(x)}{x}$$
$$= 1 \quad \leftarrow \text{l'Hôpital's rule}$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

Definition: The derivative of function at a is $\frac{d}{dx}f(x)$ or $f'(a)$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$ first principles