Today: 19/09/25

ε - δ review and practical rules for computing limits

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for every $\varepsilon > 0$, there is a $\delta > 0$ s.t if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$

* simple terms: No matter how small ε is around L, there will always be a δ around a, s.t. any x chosen will produce an $f(x)$ in the range around L

$\lim\limits_{x \to a} f(x) = L$

the ε - δ game at a and L for $f(x)$ is:
- two players A and B
- A will move first than B, then A
  - A plays (chooses) an $\varepsilon > 0$
  - B plays (chooses) an $\delta > 0$
  - A play's an x with $|x-a| < \delta$

- A wins if $|f(x) - L| \geq \varepsilon$
  ↳ not within the ε distance of L
- B wins if $|f(x) - L| < \varepsilon$
  ↳ is within the ε distance of L

$\lim\limits_{x \to a} f(x) = L$ → Winning strategy for B

* chooses δ carefully to ensure any x A picks, $f(x)$ will be within ε of L

$\lim\limits_{x \to a} f(x) \neq L$ → Winning strategy for A

* choose an ε such that no matter what δ B chooses, x will make $f(x)$ not within ε of L

Ex: $\lim\limits_{x \to 2} (3x+5) = 11$

A : $\varepsilon = 1$
B : $\delta = \frac{1}{10}$
A : needs to play an x between 1.9 and 2.1 (not 2)
$x = e - 0.7 = 2.0182\ldots$

$|3 \cdot (e - 0.7) + 5 - 11|$
$= |3 \cdot 2.0182 - 6|$
$= 0.05$

∴ B wins this time as 0.05 is less than ε

A : $\varepsilon = 1$
B : $\delta = \frac{\varepsilon}{3}$
A : x with $|x-2| < \frac{\varepsilon}{3}$

B wins if $|(3x+5) - 11| < \varepsilon$
$\iff |3x - 6| < \varepsilon$
$\iff 3|x-2| < \varepsilon$
$\iff |x-2| < \frac{\varepsilon}{3}$

$f(x)$ is continuous at a point a if...
1) $f(a)$ is defined (exists at a)
2) $\lim\limits_{x \to a} f(x)$ exists (exists as x approaches a)
3) $\lim\limits_{x \to a} f(x) = f(a)$ (the limit value equals the function value at a)

Practical Rules

0° If $f(x)$ is continuous at a, $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to 3} \frac{x^2-4x+3}{x^2-2x-3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+1)}$$

$$= \frac{3-1}{3+1} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

* basic limit equation

Types of discontinuities

↳ **Removable**: "hole" in the graph. limit exists but $f(a)$ is undefined

↳ **Jump**: function "jumps" from one point to another. limit from left and right exists but are not equal.

↳ **Vertical asymptote**: values approach positive or negative infinity as x approaches a value. limit does not exist

1° Sum Rule: $\lim_{x \to a} (f(x)+g(x)) = (\lim_{x \to a} f(x)) + (\lim_{x \to a} g(x))$

→ provided these limits exist

$$\lim_{x \to 0} (e^x + \sin(x))$$

$$= (\lim_{x \to 0} e^x) + (\lim_{x \to 0} \sin(x))$$

$$= e^0 + \sin(0) = 0 + 1 = \boxed{1}$$

2° Product Rule: $\lim_{x \to a} f(x) \cdot g(x))$

$$= (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$$

3° Constant multiple Rule: $\lim_{x \to a} cf(x) = c(\lim_{x \to a} f(x))$

4° Quotient Rule: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

can't equal zero

5° Power Rule: $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n = L^n$

6° Composition Rule: $\lim_{x \to a} f(g(x)) = \lim_{t \to b} f(t)$

where $b = \lim_{x \to a} g(x)$

Ex1: $\lim_{x\to 2} \sin(\sqrt{x^2+7}) \rightarrow \lim_{x\to 2} \sqrt{x^2+7} = \sqrt{11}$

$= \lim_{t\to\sqrt{11}} \sin(t)$

$= \boxed{\sin(\sqrt{11})}$

Ex2: $\lim_{x\to\pi} \frac{\sin(2x)}{\sin(x)} = \lim_{x\to\pi} \frac{2\cancel{\sin(x)}\cos(x)}{\cancel{\sin(x)}} = 2\cos(\pi)$

$= 2(-1) = \boxed{-2}$

Ex3: $\lim_{x\to 0} e^{(1-\cos^2(x))} = e^{\lim_{x\to 0}(1-\cos^2(x))} = e^{1-1^2} = e^0 = 1$

or

$\lim_{x\to 0} e^{\sin^2(x)} = e^{\sin^2(0)} = e^0 = 1$

or

$e^{1-\cos^2(x)} = e^{1-1^2} = e^0 = 1$ ← fast way

Ex4: $\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(1-\cos(x))}{\frac{d}{dx}x}$

l'Hôpital's Rule

↳ Don't use yet

$= \lim_{x\to 0} \frac{0-(-\sin(x))}{1}$

$= \sin(0) = 0$

method not using l'Hôpital's Rule

$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{1-\cos(x)}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)}$

$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))}$

$= \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))}$

break up and use product rule

$= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{1+\cos(x)}$

$= \left[\lim_{x\to 0} \frac{\sin(x)}{x}\right]\left[\lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)}\right]$

$\frac{\sin(0)}{1+\cos(0)} = 0$

$\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1}$

$\boxed{\lim_{x\to 0} \frac{e^x-1}{x} = 1}$

↳ remember these two

<u>Def'n</u>: The derivative of $f(x)$ at a is

$\frac{d}{dx} = f(x)$

$= f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

$x = a+h$

$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$