Today: 12/09/25

Limits and epsilonics

$\lim_{x \to a} f(x) = L$ — as x approaches a, $f(x)$ gets arbitrarily closer to L

Definition: $\lim_{x \to a} f(x) = L$

means for all $\varepsilon > 0$ ← ε = epsilonics → unkown constant
↳ must be positive *

there is a $\delta > 0$ ← Delta

such that if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$

Ex1: $f(x) = 5x+1$, $a = 12$, $L = 61$, $\varepsilon = \frac{1}{2}$

what does $\delta > 0$ have to be to ensure that $|(5x+1) - 61| < \frac{1}{2}$ whenever $|x-12| < \delta$?

← this is always delta *

$$|(5x+1) - 61| < \tfrac{1}{2}$$
$$\iff -\tfrac{1}{2} < 5x+1-61 < \tfrac{1}{2}$$
$$\iff -\tfrac{1}{2} < 5x-60 < \tfrac{1}{2}$$
$$\iff -\tfrac{1}{2} < 5(x-12) < \tfrac{1}{2}$$
$$\iff -\tfrac{1}{10} < x-12 < \tfrac{1}{10}$$
$$\iff |x-12| < \tfrac{1}{10}$$

if and only if

to get rid of 5: $\frac{1}{2} \cdot \frac{1}{5}$

$\therefore$ Any $\delta \leq \frac{1}{10}$ will work

Verification

$0 < |x-12| < 0.1$

$5|x-12| < 5(0.1)$

$|5(x-12)| < 0.5$

$|5x-60| < 0.5$

$|(5x+1) - 61| < 0.5$

✓ is reversable
↳ Reverse engineering

if ε was generic → $\varepsilon > 0$

* common question on a test

$$\iff -\varepsilon < 5x+1-61 < \varepsilon$$
$$\iff -\varepsilon < 5(x-12) < \varepsilon$$
$$\iff -\frac{\varepsilon}{5} < x-12 < \frac{\varepsilon}{5}$$
$$\iff |x-12| < \frac{\varepsilon}{5}$$

$\therefore$ Any $\delta \leq \frac{\varepsilon}{5}$

Ex2: $2x+9$, $a = 70$, $L = 149$ → $\lim_{x \to 70} (2x+9) = 149$

for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-70| < \delta$, then $|(2x+9) - 149| < \varepsilon$

$$|(2x+9) - 149| < \varepsilon$$
$$\iff |2x-140| < \varepsilon$$
$$\iff |2(x-70)| < \varepsilon$$
$$\iff |2| \cdot |x-70| < \varepsilon$$
$$\iff 2|x-70| < \varepsilon$$
$$\iff |x-70| < \frac{\varepsilon}{2}$$

← didn't drop $|x|$ absolute value, so it can only be positive

if you dake a negative number out of an absolute value it becomes positive

$\therefore$ So any $\delta \leq \frac{\varepsilon}{2}$ works
↳ since every step is reversible *

Ex3: $-3x + \frac{1}{2}$, $a = 2$, $L = -5.5$ → $\lim_{x \to 2} |(-3x + \frac{1}{2}) - (-\frac{11}{2})|$

for any $\varepsilon > 0$, there is a $\delta > 0$, so if $|x-2| < \delta$, then $|(-3x + \frac{1}{2}) - (-\frac{11}{2})| < \varepsilon$

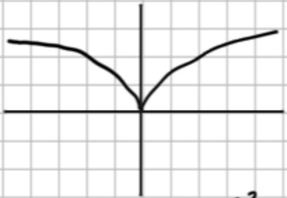
$$\iff |-3x + \tfrac{1}{2} + \tfrac{11}{2}| < \varepsilon$$
$$\iff |-3x + \tfrac{12}{2}| < \varepsilon$$
$$\iff |-3x+6| < \varepsilon$$
$$\iff |-3| \cdot |x-2| < \varepsilon$$
$$\iff 3|x-2| < \varepsilon$$
$$\iff |x-2| < \frac{\varepsilon}{3}$$

$\therefore$ So any $\delta \leq \frac{\varepsilon}{3}$ works,

Say this on a test → * because every step is reversible

Ex 4: $\sqrt{|x|}$ $\lim_{x\to 0}\sqrt{|x|}=0$

For every $\varepsilon>0$, there is a $\delta>0$, such that if $|x-0|<\delta$, then $|\sqrt{|x|}-0|=\varepsilon$



$$|\sqrt{|x|}-0|<\varepsilon$$
$$\Leftrightarrow |\sqrt{|x|}|<\varepsilon$$
$$\Leftrightarrow \sqrt{|x|}<\varepsilon$$
$$\Leftrightarrow (\sqrt{|x|})^2<\varepsilon^2$$
$$\Leftrightarrow |x|=\varepsilon^2$$
$$|x-0|=\varepsilon^2$$

∴ So $\delta\le\varepsilon^2$ does the job, because it's reversable

if you don't say this you can lose points

*quadratics are slightly harder

s.t. = such that

Ex 5: x^2, $a=9$ $\lim_{x\to 9}x^2=81$

for each $\varepsilon>0$ there is a $\delta>0$ s.t. if $|x-9|<\delta$ then $|x^2-81|<\varepsilon$

$$|x-9|<\delta\le 1$$
$$-1\le x-9\le 1$$
$$8\le x\le 10$$
$$17\le x+9\le 19$$
$$17\le |x+9|\le 19$$
$$\frac{1}{17}\le\frac{1}{|x+9|}\le\frac{1}{19}$$

we will accept no $\delta>1$

$\delta\le 1$

$$|x^2-81|<\varepsilon$$
$$|(x+9)(x-9)|<\varepsilon$$
$$|x+9|\cdot|x-9|<\varepsilon$$
$$|x-9|<\frac{1}{19}\varepsilon\le\cancel{\frac{\varepsilon}{|x+9|}}$$

can't have x there

*putting on an initial restriction on δ

↳ $\delta\le 1\rightarrow |x-9|<\delta\le 1$

this implies: $-1<x-9<1$ (add 9 to all parts)

$8<x<10$

$17<|x+9|<19$ (adding 9 to everything)

∴ $|x+9|$ is always between the intervals 17 and 19 when $\delta\le 1$

next step is making it's reciprical to put into $|x-9|<\varepsilon\cdot\frac{1}{|x+9|}$

when doing reciprical 19 and 17 switches

↳ $\frac{1}{19}$ is less than $\frac{1}{17}$

$$\boxed{\frac{1}{19}<\frac{1}{|x+9|}}<\frac{1}{17}$$

choose the smaller bound to guarantee that "a" is within the intervals with ε of the limit

↳ $|x-9|<\varepsilon\cdot\frac{1}{|x+9|}$

$$|x-9|<\varepsilon\cdot\frac{1}{19}$$
$$|x-9|<\frac{\varepsilon}{19}$$

∴ So any $\delta\le\frac{\varepsilon}{17}$ works, because every step is reversible