Lecture #3-calculus 1

Friday, September 12, 2025 9:11 AM

Limits and the horror of epsilonics

$\lim_{x \to a} f(x) = L$ means... as x gets arbitrarily close to a $f(x)$ gets arbitrarily close to L

x = variable

a, L = numbers

$f(x)$ = Function

Definition: $\lim_{x \to a} F(x) = L$

means for all $\varepsilon > 0$ (epsilon; must be +ve)
there is $\delta > 0$ (delta) such that (for all x)
if $|x-a| < \delta$, then $|f(x) - L| <$ (absolute value)

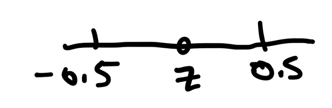
Example:
$f(x) = 5x+1$
$a = 12$ $L = 61$
$\varepsilon = \frac{1}{2}$ or 0.5

What does $\delta > 0$ have to be to ensure that $|(5x+1) - 61| < \frac{1}{2}$ whenever $(x-12) < \delta$?

absolute value ex:

$|z| < \frac{1}{2}$

$\hookrightarrow -\frac{1}{2} < z < \frac{1}{2}$



$|(5x+1) - 61| < \frac{1}{2}$ (→L)

$\Leftrightarrow -\frac{1}{2} < 5x+1-61 < \frac{1}{2}$

① $\Leftrightarrow -\frac{1}{2} < 5x - 60 < \frac{1}{2}$

② $\Leftrightarrow -\frac{1}{2} < 5(x-12) < \frac{1}{2}$

③ $\Leftrightarrow -\frac{1}{2} \cdot \frac{1}{5} < x - 12 < \frac{1}{2} \cdot \frac{1}{5}$

$-\frac{1}{10} = -\frac{1}{2} \cdot \frac{1}{5} < x - 12 < \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$

$\therefore \Leftrightarrow |x-12| < \frac{1}{10}$ any $\delta \le \frac{1}{10}$ will work

1. simplify 1-61
2. factor a 5 out of 5x-60
3. divide everything by 5

Another example

$g(x) = 2x+9$
$a = 70$ (2×2)
$L = 149$

check: $\lim_{x \to 70} (2x+9) = 149$

for any $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x-70| < \delta$, then $|(2x+9) - 149| < \varepsilon$

if and only if

$|(2x+9) - 149| < \varepsilon$

$\Leftrightarrow |2x - 140| < \varepsilon$

$\Leftrightarrow |2(x-70)| < \varepsilon$

$\Leftrightarrow |2| \cdot |x-70| < \varepsilon$ } not going to worry abt

$\Leftrightarrow 2|x-70| < \varepsilon$

$\Leftrightarrow |x-70| < \frac{\varepsilon}{2}$ *so any $\delta \le \frac{\varepsilon}{2}$ works

since every step is reversible

Another example

$h(x) = -3x + \frac{1}{2}$
$a = 2$
$L = -5.5 = -\frac{11}{2}$

$\lim_{x \to 2} |(-3x + \frac{1}{2}) - (\frac{11}{2})|$

For any $\varepsilon > 0$, there is a $\delta > 0$ s.t. if $|x-2| < \delta$, then $|(-3x+\frac{1}{2}) - (-\frac{11}{2})| < \varepsilon$

(or iff, or ≡)

$|(-3x + \frac{1}{2}) - (-\frac{11}{2})| < \varepsilon$

$\Leftrightarrow |-3x + \frac{1}{2} + \frac{11}{2}| < \varepsilon$

$\Leftrightarrow |-3x + \frac{12}{2}| < \varepsilon$

$\Leftrightarrow |-3x + 6| < \varepsilon$

$\Leftrightarrow |(-3)(x-2)| < \varepsilon$

$\Leftrightarrow 3|x-2| < \varepsilon$

$\Leftrightarrow |x-2| < \frac{\varepsilon}{3}$

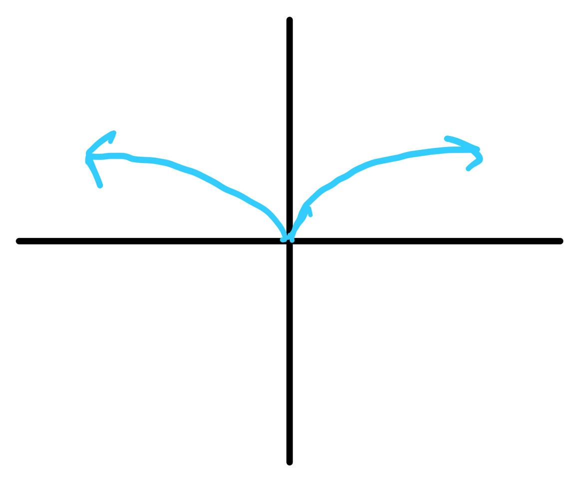
Thus $\delta \ge \frac{\varepsilon}{3}$ works, because every step is reversible

↳ This applies to linear funcs mostly. For non-linear funcs, most of the time steps will be irreversible.

Another example

$h(x) = \sqrt{|x|}$

$\lim_{x \to 0} \sqrt{|x|} = 0$



For every $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x-0| < \delta$, then $|\sqrt{|x|} - 0| < \varepsilon$

$|\sqrt{|x|} - 0| < \varepsilon$

$\Leftrightarrow |\sqrt{|x|}| < \varepsilon$

$\Leftrightarrow (\sqrt{|x|})^2 < \varepsilon^2$

(where $(\sqrt{|x|})^2 = |x|$)

$\Leftrightarrow |x-0| < \varepsilon^2$

$\delta \le \varepsilon^2$ ∴ does the job

Another example

$p(x) = x^2$
$a = 9$

$\lim_{x \to 9} x^2 = 81$

for each $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x-9| < \delta$, then $|x^2 - 81| < \varepsilon$

δ cannot depend on x bc it is used to control x

$|x^2 - 81| < \varepsilon$

$\Leftrightarrow |(x-9)(x+9)| < \varepsilon$

$\Leftrightarrow |x-9| \cdot |x+9| < \varepsilon$

$\Leftrightarrow |x-9| < \frac{\varepsilon}{|x+9|}$ (crossed out) → We will accept no $\varepsilon > 1$ ↳ i.e. $\varepsilon \le 1$

$\Leftrightarrow |(x+9)(x-9)| \le 1$

$|x^2 - 81|$ → $|x^2 - 81| \le 1$

$\Leftrightarrow -1 \le x^2 - 81 \le 1$

$\Leftrightarrow 80 \le x^2 \le 82$ since $x > 0$ near $a = 9$

$\Leftrightarrow \sqrt{80} \le x \le \sqrt{82}$

$\Leftrightarrow 9 + \sqrt{80} \le x + 9 \le 9 + \sqrt{82}$

$\Leftrightarrow \frac{1}{9+\sqrt{82}} \le \frac{1}{x+9} \le \frac{1}{9+\sqrt{80}}$

$\Leftrightarrow \frac{1}{9+\sqrt{82}} \le \frac{1}{|x+9|} \le \frac{1}{9+\sqrt{80}}$ } ?

$\Leftrightarrow |x-9| < \delta \le 1$

$\Leftrightarrow -1 \le x - 9 \le 1$

$\Leftrightarrow 8 \le x \le 10$

$\Leftrightarrow 17 \le x + 9 \le 19$

$\Leftrightarrow 17 \le |x+9| \le 19$

$\Leftrightarrow \frac{1}{19} \le \frac{1}{(x+9)} \le \frac{1}{17}$

$\delta = \min(1, \frac{\varepsilon}{19})$