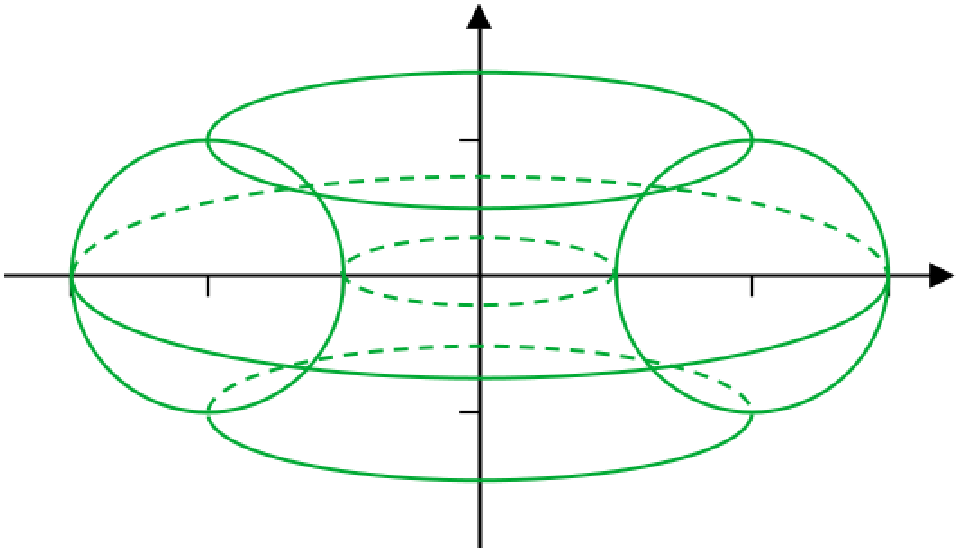
Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2024

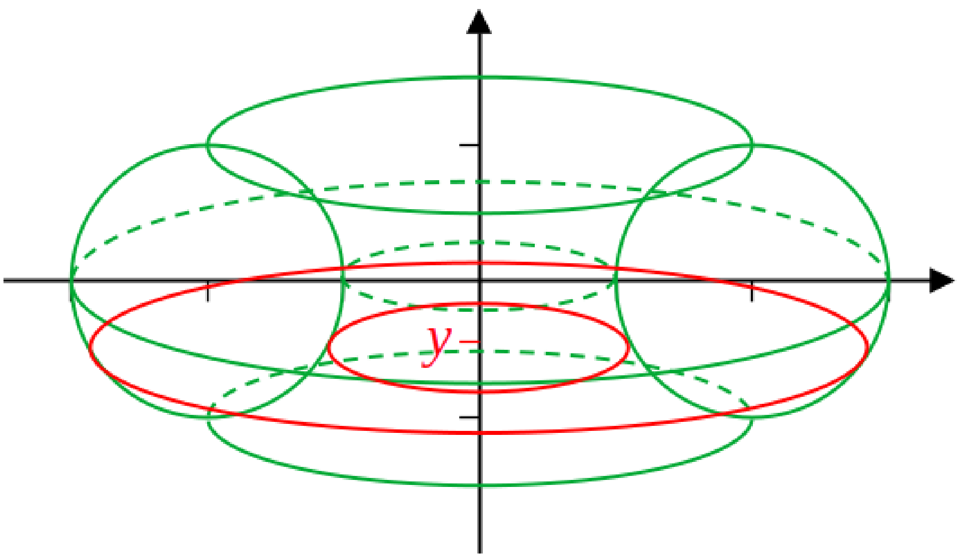
Solutions to Assignment #6
How Much Doughnut?

Consider the torus[†] obtained by revolving the circle $(x-2)^2+y^2=1$ about the y-axis.



1. Compute the volume of this torus. *[10]*

Solution 1. *Disk/washer method.* Since the axis of revolution is the y-axis, the cross-sections are washers stacked vertically, so we use y as the basic variable. Note that $-1 \leq y \leq 1$ for the original region, and that if $(x-2)^2+y^2=1$, then $x = 2 \pm \sqrt{1-y^2}$.



† Basically, mathematician-speak for "doughnut". :-)

The washer at y has outer radius $R = x - 0 = 2 + \sqrt{1-y^2}$ and inner radius $r = x - 0 = 2 - \sqrt{1-y^2}$, and hence has area

$$\begin{aligned} A(y) &= \pi\left[R^2 - r^2\right] = \pi\left[\left(2+\sqrt{1-y^2}\right)^2 - \left(2-\sqrt{1-y^2}\right)^2\right] \\ &= \pi\left[\left(4+4\sqrt{1-y^2}+1-y^2\right) - \left(4-4\sqrt{1-y^2}+1-y^2\right)\right] = 8\pi\sqrt{1-y^2}. \end{aligned}$$

The volume of the torus is therefore given by

$$V = \int_{-1}^{1} A(y)\,dy = \int_{-1}^{1} 8\pi\sqrt{1-y^2}\,dy.$$

This is an integral we do not how to do with the techniques we have learned so far – wait for trigonometric substitutions in MATH 1120H – so we hand it off to SageMath:

```
[1]: var('y')
     integral( 8*pi*sqrt(1-y^2), y, -1, 1 )

[1]: 4*pi^2
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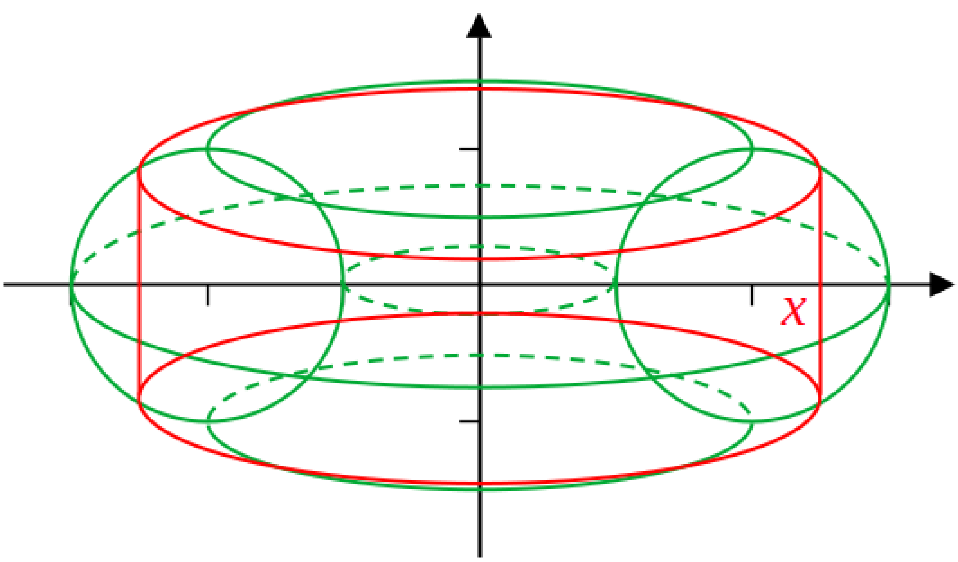
The volume of the torus is therefore $4\pi^2$. □

SOLUTION 2. *Cylindrical shell method.* Since the axis of revolution is the y-axis, the cylindrical shells are parallel to the y-axis and perpendicular to the x-axis, so we use x as the basic variable. Note that $1 \le x \le 3$ for the original region, and that if $(x-2)^2+y^2 = 1$, then $y = \pm\sqrt{1-(x-2)^2}$.



The cylinder at x has radius $r = x - 0 = x$ and height $h = \sqrt{1-(x-2)^2} - \left(-\sqrt{1-(x-2)^2}\right) = 2\sqrt{1-(x-2)^2}$, and hence has area

$$A(x) = 2\pi rh = 2\pi x 2\sqrt{1-(x-2)^2} = 4\pi x\sqrt{1-(x-2)^2}.$$

The volume of the torus is therefore given by

$$V = \int_1^3 2\pi rh\,dx = \int_1^3 4\pi x\sqrt{1-(x-2)^2}\,dx.$$

Rather than spend time trying to figure out how to compute this integral – see what you have after the substitution $u = x - 2$ – we again hand off to SageMath to do the heavy lifting:

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[2]: integral( 4*pi*x*sqrt(1-(x-2)^2), x, 1, 3 )
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[2]: 4*pi^2
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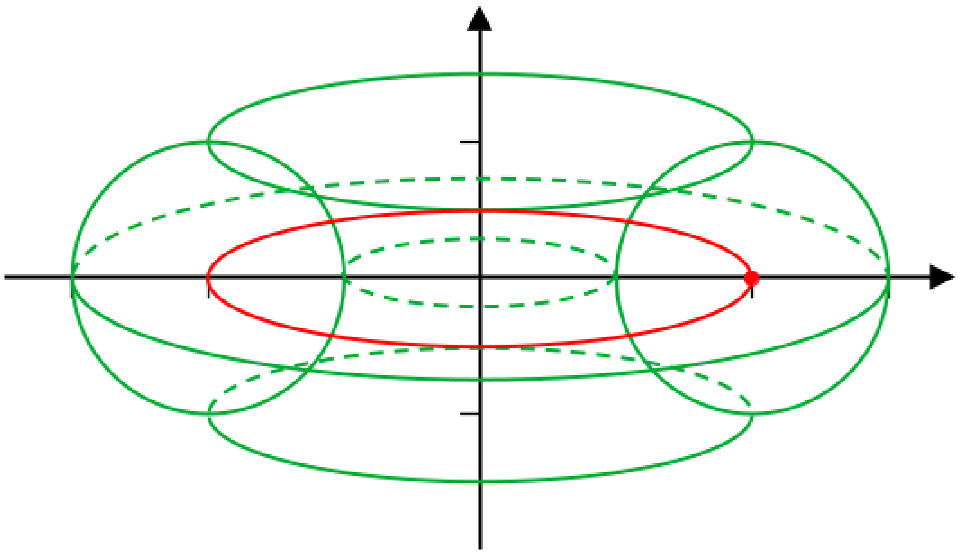
Again, the volume of the torus is therefore $4\pi^2$. □

Solution 3. *Using Pappus' Centroid Theorem.* The *centroid* of a two-dimensional region is the centre of mass of the region if it was a thin plate of uniform thickness and density. Because of its symmetry, the centroid of a circle or disk is just the centre of the circle.

The theorem in question asserts that the volume of a solid of revolution can be computed by multiplying the area of the region being revolved by the circumference of the circle the centroid of that region moves along when the region is revolved.



The centre of the original region enclosed by the circle $(x-2)^2 + y^2 = 1$ is the point $(2, 0)$. When revolved about the y-axis, this point moves along a circle of radius $R = 2$, which must have perimeter $2\pi R = 4\pi$. The region itself has radius $r = 1$, and hence area $\pi r^2 = \pi$. By Pappus' Centroid Theorem, it follows that the volume of the torus is $4\pi \cdot \pi = 4\pi^2$; in general, a torus with internal radius r and centroid radius R has area $2\pi R \cdot \pi r^2 = 2\pi^2 R r^2$. ■

Note. Pappus of Alexandria, for whom we have no firm dates of birth or death, but who seems to have been active in the first half of the 4th Century AD, was the last of the great mathematicians in the classical Greek and Hellenistic tradition. His major mostly surviving work is the *Collection*, which attempts to summarize and comment on a large part of Greek and Hellenistic mathematics, as well as adding results of his own. His centroid theorems – there is another dealing with surface areas of solids of revolution – appear in Book VII of the *Collection*.