Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals

Trent University, Fall 2024

Assignment #5
Definite Integrals via the Right-Hand Rule

A concept and a bit of notation first. The *definite integral* from a to b of $f(x)$, usually denoted by $\int_a^b f(x)\,dx$, is the weighted area betwen the graph of $y = f(x)$ for x between a and b and the x-axis. "Weighted area" means that area above the x-axis is added and area below the x-axis is subtracted. Definite integrals are usually computed using antiderivatives, but they need to be defined in some other way first. There are also cases where we need to compute a definite integral, or at least approximate it, but the function in question has no antiderivative.

Before tackling this assignment, please read the accompanying handout, *Right-Hand Rule Riemann Sums*, which describes a still-somewhat-useful simplification of the defonition of a definite integral. (If you have an interest in seeing a full-fat definition of the definite integral, which is beyond the scope of this course, check out the handout *A Precise Definition of the Definite Integral*.) If you haven't already seen them in – or have forgotten them from – Assignment #3, please look up SageMath's `sum` and `limit` commands before tackling this assignment. This assignment will also ask you to use SageMath's `integral` command, which can be used to compute definite integrals as well as find "indefinite integrals" (*i.e.* generic antiderivatives).

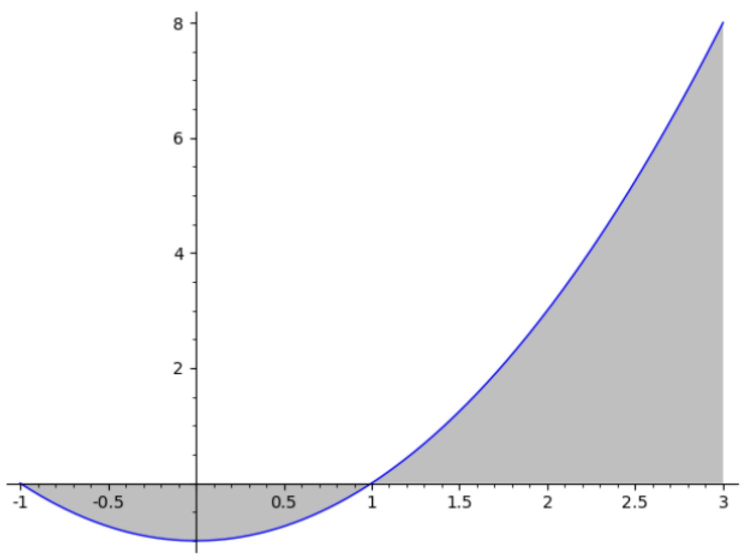
Consider the definite integral $\int_{-1}^{3} \left(x^2 - 1\right)\, dx$.

1. Sketch the region whose weighted area is computed by this definite integral. *[1]*

Solution. Cheating just a bit, we use SageMath:

```
[1]: plot( x^2 - 1, -1, 3, fill=True )
```

[1]:



□

2. Set up the Right-Hand Rule formula – a limit of a sum as given on page 2 of the handout *Right-Hand Rule Riemann Sums* – for computing this definite integral. *[2]*

SOLUTION. The generic Right-Hand Rule formula for a definite integral $\int_a^b f(x)\,x$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i \cdot \frac{b-a}{n}\right) \right].$$

In the present case, we have $a = -1$, $b = 3$, and $f(x) = x^2 - 1$, so the Right-Hand Rule formula works out to be

$$\int_{-1}^{3} \left(x^2 - 1\right)\,dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{3-(-1)}{n} \cdot \left(\left(-1 + i \cdot \frac{3-(-1)}{n}\right)^2 - 1 \right) \right]$$

before any simplification. □

3. Evaluate the formula you obtained in solving question **2** using SageMath. *[2]*

SOLUTION. Rather than type in the messy formula given above for **2**, we type a code fragment into SageMath that uses the generic Right-Hand Rule formula and is adaptable to other functions and intervals:

```
[2]: var('n')
     var('i')
     f = function('f')(x)
     a = -1
     b = 3
     f(x) = x^2 - 1
     limit( sum( (b-a)/n * f( a + i*(b-a)/n ), i, 1, n ), n = oo )

[2]: 16/3
```

□

4. Evaluate the formula you obtained in part **b** by hand. *[4]*

Hint: Some of the summation formulas you obtained in Assignment #4 are likely be useful here.

SOLUTION. We will use two of the summation formulas from Assignment #4, namely

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$. Here we go:

$$\begin{aligned}
\int_{-1}^{3} \left(x^2 - 1\right) dx &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{3-(-1)}{n} \cdot \left(\left(-1 + i \cdot \frac{3-(-1)}{n} \right)^2 - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{4}{n} \cdot \left(\left(-1 + i \cdot \frac{4}{n} \right)^2 - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left(\left(-1 + \frac{4i}{n} \right)^2 - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left((-1)^2 - 2 \cdot \frac{4i}{n} + \left(\frac{4i}{n} \right)^2 - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left(1 - \frac{8i}{n} + \frac{16i^2}{n^2} - 1 \right) \right] = \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left(\frac{-8i}{n} + \frac{16i^2}{n^2} \right) \right] \\
&= \lim_{n\to\infty} \frac{4}{n} \cdot \left[\left(-\sum_{i=1}^{n} \frac{8i}{n} \right) + \left(\sum_{i=1}^{n} \frac{16i^2}{n^2} \right) \right] \\
&= \lim_{n\to\infty} \frac{4}{n} \cdot \left[\left(-\frac{8}{n} \cdot \sum_{i=1}^{n} i \right) + \left(\frac{16}{n^2} \sum_{i=1}^{n} i^2 \right) \right] \\
&= \lim_{n\to\infty} \frac{4}{n} \cdot \left[-\frac{8}{n} \left(\frac{n^2}{2} + \frac{n}{2} \right) + \frac{16}{n^2} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \right] \\
&= \lim_{n\to\infty} \frac{4}{n} \cdot \left[-4n - 4 + \frac{16n}{3} + 8 + \frac{8}{3n} \right] = \lim_{n\to\infty} \frac{4}{n} \cdot \left[\frac{4n}{3} + 4 + \frac{8}{3n} \right] \\
&= \lim_{n\to\infty} \left[\frac{16}{3} + \frac{16}{n} + \frac{32}{3n^2} \right] = \frac{16}{3} + 0 + 0 = \frac{16}{3} \quad \square
\end{aligned}$$

5. Evaluate the given definite integral using SageMath's `integral` command. *[1]*

SOLUTION. Here we are:

```
[3]: integral( f(x), x, -1, 3 )
```

```
[3]: 16/3
```

■