Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2024

Solutions to Assignment #2
Sagely Analysis

1. Consider the quadratic function $f(x) = x^2 - 5x + 6$.

a. Use SageMath to plot $y = f(x)$ for $0 \leq x \leq 5$. Use the graph to guess at the roots of $y = f(x)$. *[1]*

b. Use SageMath to find the roots of $y = f(x)$ by solving the equation $f(x) = 0$. *[1]*

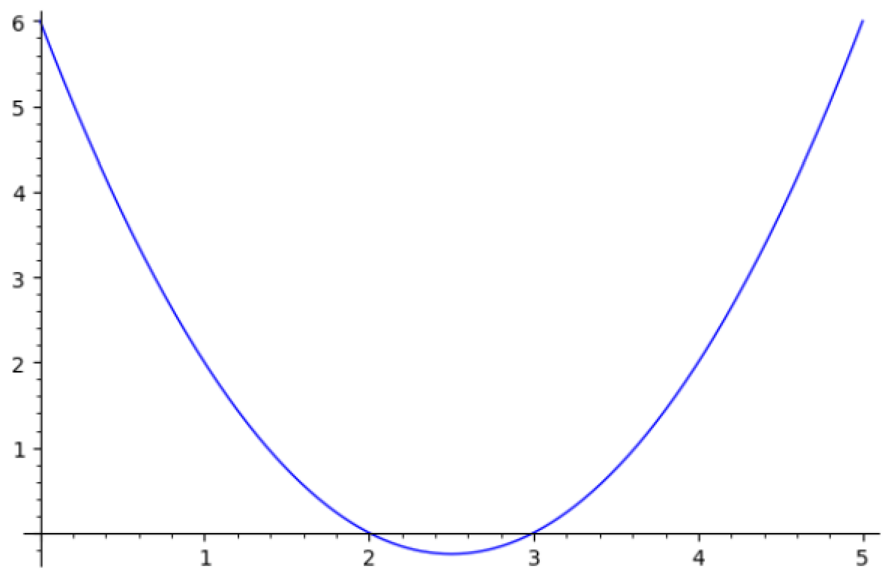
c. Use the quadratic formula by hand to find the roots of $y = f(x)$. *[0.5]*

SOLUTION. **a.** Here we go!

[2]:
```
# MATH 1110H-A Fall 2024
# The SageMath part of the Solutions to Assignment #2

# 1a
f = function('f')(x)
f(x) = x^2 - 5*x + 6
plot( f(x), 0, 5 )
```

[2]:



It's a pretty good guess from this graph that the roots of $f(x)$ are $x = 2$ and $x = 3$.

□

b. Here it is:

[3]:
```
# 1b
solve( f(x) == 0, x )
```

[3]: `[x == 3, x == 2]`

Thus the roots of $f(x) = x^2 - 5x + 6$ are $x = 2$ and $x = 3$. $\square$

c. Recall that the quadratic formula tells us that the roots of a generic quadratic $p(x) = ax^2 + bx + c$, where $a \neq 0$, are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In the case of $f(x) = x^2 - 5x + 6$ we have $a = 1$, $b = -5$, and $c = 6$; plugging these into the formula tells us that the roots of $f(x)$ are:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2}$$

That is, the roots of $f(x)$ are $x = \frac{5-1}{2} = \frac{4}{2} = 2$ and $x = \frac{5+1}{2} = \frac{6}{2} = 3$. $\blacksquare$

2. Consider the quartic function $g(x) = x^4 - 10x^3 + 35x^3 - 50x + 24$.

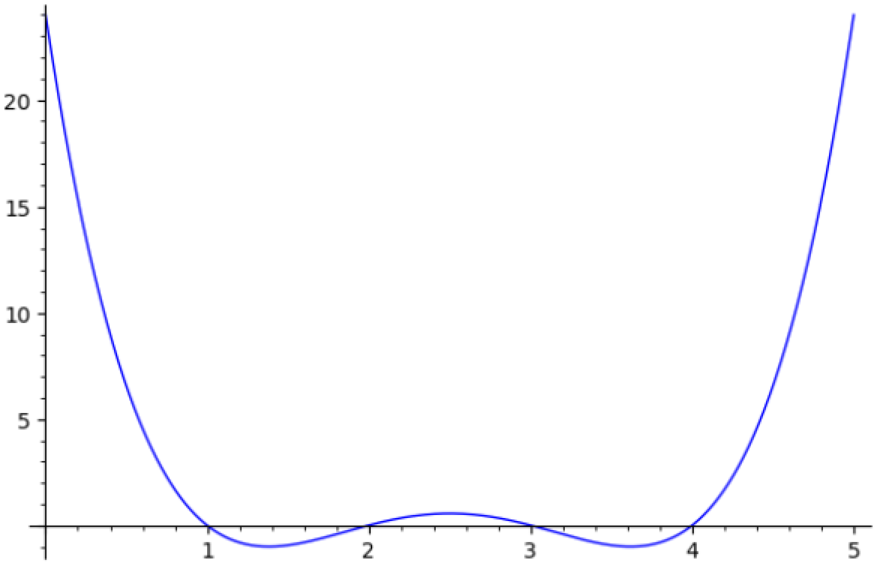
- **a.** Use SageMath to plot $y = g(x)$ for $0 \leq x \leq 5$. Use the graph to guess at the roots of $y = g(x)$. *[1]*
- **b.** Use SageMath to find the roots of $y = g(x)$ by solving the equation $g(x) = 0$. *[1]*
- **c.** What polynomial is the function $h(x) = g(x)/f(x)$ equal to, except, of course, when $f(x) = 0$? *[0.5]*
- **d.** Use the quartic formula by hand to find the roots of $y = g(x)$. *[Bonus = 1]*

Note. The bonus mark for part **d** will be hard-earned if you choose to try . . .

Solution. **a.** Onwards!

[4]:
```
# 2a
g = function('g')(x)
g(x) = x^4 - 10*x^3 + 35*x^2 - 50*x + 24
plot( g(x), 0, 5 )
```

[4]:



It's a pretty good guess from this graph that the roots of $g(x)$ are $x = 1$, $x = 2$, $x = 3$, and $x = 4$. $\square$

b. Here it is:

```
[5]: # 2b
solve( g(x) == 0, x )
```

Thus the roots of $g(x) = x^4 - 10x^3 + 35x^3 - 50x + 24$ are $x = 1$, $x = 2$, $x = 3$, and $x = 4$. □

c. Well, $f(x) = x^2 - 5x + 6 = (x-2)(x+3)$ and $g(x) = x^4 - 10x^3 + 35x^3 - 50x + 24 = (x-1)(x-2)(x-3)(x-4)$. (Multiply things out to check!) It follows that, at least when $f(x) \neq 0$, that:

$$h(x) = \frac{g(x)}{f(x)} = \frac{(x-1)(x-2)(x-3)(x-4)}{(x-2)(x-3)} = (x-1)(x-4) = x^2 - 5x + 4 \quad \square$$

d. If you've actually looked up the quartic formula, you'll understand why I'm not giving a solution here . . . ■

3. **a.** Use SageMath to plot $y = x$ and $y = \sin(x)$ together for $-\pi \leq x \leq \pi$. *[1]*

b. Use SageMath to plot $y = |x|$ and $y = |\sin(x)|$ together for $-\pi \leq x \leq \pi$. *[1]*

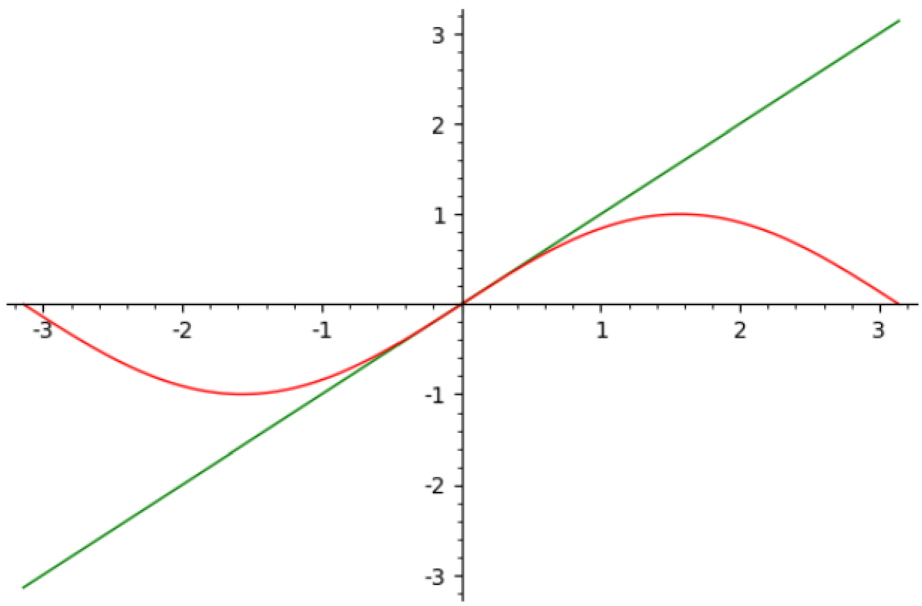
c. Use plots drawn by SageMath to make an argument that $|\sin(x)| \leq |x|$ for all x, and that equality occurs only when $x = 0$. *[1]*

NOTE. For part **c**, it might help to draw some additional plots for other ranges of x.

SOLUTION. **a.** Here be the plot:

```
[7]: # 3a
plot( x, -pi, pi, color='green' ) + plot( sin(x), -pi, pi, color='red' )
```

[7]:

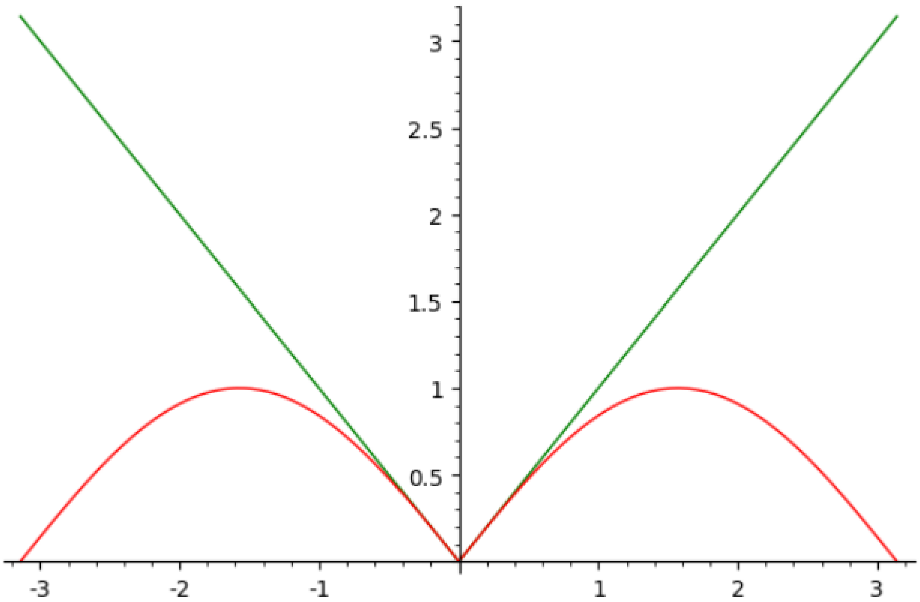


□

b. The plotting continues:

[8]:
```
# 3b
plot( abs(x), -pi, pi, color='green' ) + plot( abs(sin(x)), -pi, pi,␣
↪color='red' )
```

[8]:

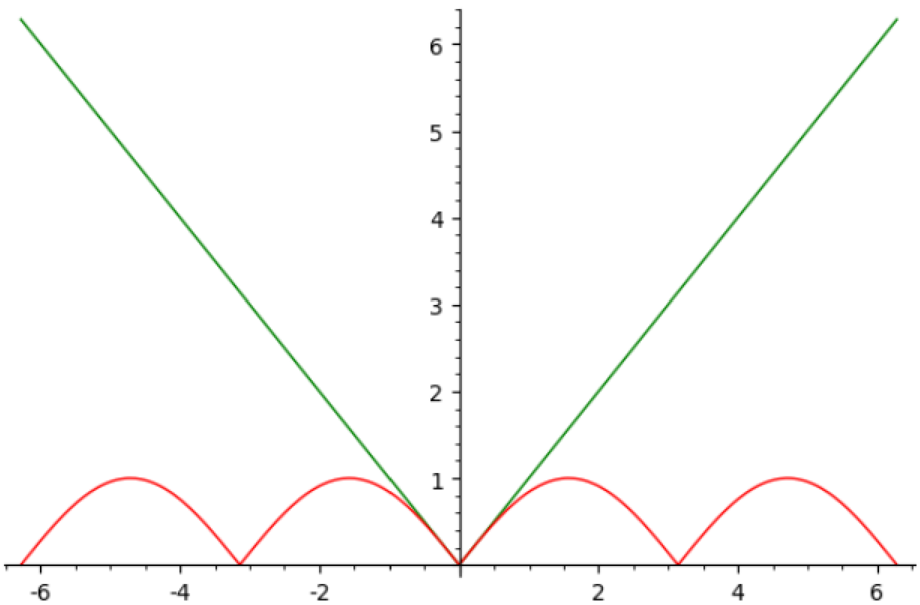


□

c. Along with the plot for part **b** above, consider the following two plots of $y = |x|$ and $y = |\sin(x)|$ for different ranges of x, one for $-2\pi \leq x \leq 2\pi$:

[9]:
```
# 3c
plot( abs(x), -2*pi, 2*pi, color='green' ) + plot( abs(sin(x)), -2*pi, 2*pi,␣
↪color='red' )
```

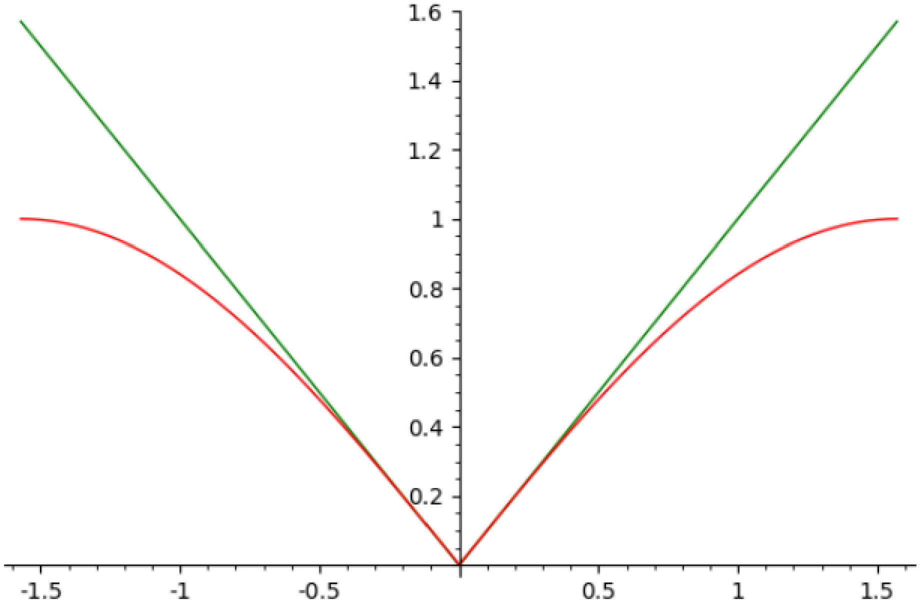
[9]:



… and one for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$:

[10]:
```
plot( abs(x), -pi/2, pi/2, color='green' ) + plot( abs(sin(x)), -pi/2, pi/2,␣
↪color='red' )
```

[10]:



At each scale the graph of $y = |x|$ is above the graph for $y = |\sin(x)|$, at least until you get close to $x = 0$. Near $x = 0$ the graphs get very close to each other … This provides some evidence, though not entirely conclusive evidence, that $|\sin(x)| \leq |x|$ for all x, except maybe for x very close to 0. ■

4. Use SageMath to find the inverse function of $s(x) = \dfrac{e^x - e^{-x}}{2}$. *[2]*

Hint: $t(x)$ is the inverse function of $s(x)$ if $y = t(x)$ exactly when $x = s(y)$. Also, ask yourself whether the answers SageMath gave you make sense.

Solution. Following the hint:

[12]:
```
# 4
var('y')
s = function('s')(y)
s(y) = ( e^y - e^(-y) ) / 2
solve( x == s(y), y )
```

[12]: `[y == log(x - sqrt(x^2 + 1)), y == log(x + sqrt(x^2 + 1))]`

SageMath actually gives us two possible inverses for $s(x) = \dfrac{e^x - e^{-x}}{2}$, namely $y = \ln\left(x - \sqrt{x^2+1}\right)$ and $y = \ln\left(x + \sqrt{x^2+1}\right)$. However, since $\ln(t)$ is defined only for $t > 0$

and $x - \sqrt{x^2+1} < 0$ for all x (Why?), the first "inverse" given by SageMath is actually undefined. The second one is well-defined because $x + \sqrt{x^2+1} > 0$ for all x (Why?). Problems like this are one reason to do sanity checks on what symbolic computations produce, whether these are done by hand or by software. ■

NOTE. The function $s(x)$ in question **4** is usually called $\sinh(x)$ (pronounced "sinch"), and it is one of the basic hyperbolic functions, along with $\cosh(x) = \dfrac{e^x - +e^{-x}}{2}$ (pronounced "kosh"). The hyperbolic functions turn up as auxiliary trigonometric functions in hyperbolic space, as solutions to differential equations, and various other places in mathematics, both pure and applied. They are related to both the exponential functions (Duh!) and the usual trigonomteric functions.