Mathematics 1110H (Section B) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023

Solutiions to Quiz #6
A Little Optimization

REMINDER. While you are allowed to work together and look things up when doing the quizzes and assignments, your submission should be written up entirely by yourself, giving credit to any collaborators or sources that you ended up actually using. *Please show all your steps and simplify your answers as far as practical.*

Let $f(x) = \dfrac{-x}{1+x^2}$.

0. What is the domain of $f(x)$, *i.e.* for which values of x is $f(x)$ defined? *[0.5]*

SOLUTION. Since $1 + x^2 \geq 1$ for all x, the denominator of $f(x) = \dfrac{-x}{1+x^2}$ is never 0, so $f(x)$ is defined for all x. That is, the domain of $f(x)$ is "all x", or $\mathbb{R}$, or $(-\infty, \infty)$, or "all real numbers", or … $\square$

1. Explain why $-1 < f(x) < 1$ for all values of x. *[0.5]*

SOLUTION. Since the square root function is an increasing function, we have $\sqrt{1+x^2} > \sqrt{x^2} = |x|$ for all x, from which it follows that $-\sqrt{1+x^2} < -x < \sqrt{1+x^2}$ for all x. This, in turn, means that $-1 < \dfrac{-x}{1+x^2} < 1$, *i.e.* $-1 < f(x) < 1$ for all x. $\square$

2. Find $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to \infty} f(x)$. *[1]*

SOLUTION. One could use l'Hôpital's Rule, but it's just as easy to use a little algebra to evaluate the limits:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{-x}{1+x^2} = \lim_{x \to -\infty} \frac{-x}{1+x^2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to -\infty} \frac{-1}{\frac{1}{x} + x} \begin{matrix} \to -1 \\ \to 0 - \infty \end{matrix} = 0^+$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{-x}{1+x^2} = \lim_{x \to \infty} \frac{-x}{1+x^2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{-1}{\frac{1}{x} + x} \begin{matrix} \to -1 \\ \to 0 + \infty \end{matrix} = 0^-$$

Recall that 0^- and 0^+ indicate that 0 is being approached through negative numbers and through positive numbers, repsectively. $\square$

3. Find $f'(x)$. *[1]*

SOLUTION. We'll use the Quotient and Power Rules:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{-x}{1+x^2}\right) = \frac{\left[\frac{d}{dx}(-x)\right]\left(1+x^2\right) - (-x)\left[\left(1+x^2\right)\right]}{\left(1+x^2\right)^2} \\ &= \frac{[-1]\left(1+x^2\right) + x\,[2x]}{\left(1+x^2\right)^2} = \frac{-1 - x^2 + 2x^2}{\left(1+x^2\right)^2} = \frac{x^2 - 1}{\left(1+x^2\right)^2} \quad \square \end{aligned}$$

4. Find any and all critical points of $f(x)$. *[0.5]*

Solution. The critical points of $f(x)$ occur when $f'(x) = 0$ or is undefined. Since $1 + x^2 \geq 1 > 0$ for all x, $f'(x) = \dfrac{x^2 - 1}{(1 + x^2)^2}$ is defined for all x, so it remains only to find those points where $f'(x) = 0$.

$$f'(x) = \frac{x^2 - 1}{(1 + x^2)^2} = 0 \iff x^2 - 1 = 0 \iff x^2 = 1 \iff x = \pm 1$$

Thus the critical points of $f(x)$ occur when $x = -1$ and when $x = 1$. □

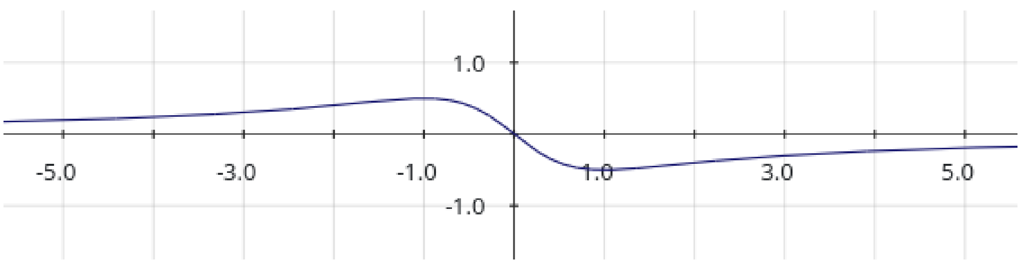
5. Find the absolute maximum and minimum values of $f(x)$. *[1.5]*

Solution. Recall from **0** that $f(x) = \dfrac{-x}{1 + x^2}$ is defined on all of $(-\infty, \infty)$; since it is a rational function it is continuous and differentiable wherever it is defined. To find the absolute maximum and minimum of $f(x)$ we therefore need only compare its values – in this case its limits – at the endpoints $x = \pm\infty$ of the interval and at its critical points $x = \pm 1$: the largest will be the absolute maximum and the smallest will be the absolute minimum.

$$\begin{aligned}
\lim_{x \to -\infty} f(x) &= 0^+ \\
\lim_{x \to \infty} f(x) &= 0^- \\
f(-1) &= \frac{-(-1)}{1 + (-1)^2} = \frac{1}{2} \\
f(1) &= \frac{-1}{1 + 1^2} = -\frac{1}{2}
\end{aligned}$$

Thus $f(x) = \dfrac{-x}{1 + x^2}$ has absolute maximum $f(-1) = \dfrac{1}{2}$ and absolute minimum $f(1) = -\dfrac{1}{2}$.
■

Sanity Check. We plot $f(x) = \dfrac{-x}{1 + x^2}$ to see if the graph agrees with the above:



Seems about right!