Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023

Solutions to Assignment #5
Definite integrals done the hard – but not hardest! – way!

Warning: Please read the accompanying handout *Right-Hand Rule Riemann Sums* for the necessary definitions and a simple example.

1. Use the Right-Hand Rule to compute $\displaystyle\int_{-1}^{3} \left(x^2 - 1\right)\, dx$ by hand. *[6]*

You may find the summation formulas $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ to be useful in working through **1**.

SOLUTION. The Right-Hand Rule formula for computing the definite integral $\displaystyle\int_a^b f(x)\, dx$ is:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \left[\frac{b-a}{n} \cdot \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right)\right]$$

We plug $a = -1$, $b = 3$, and $f(x) = x^2 - 1$ into this formula – with a slight delay for $f(x)$ to simplify other bits first – and then go to work:

$$\begin{aligned}
\int_{-1}^{3} \left(x^2 - 1\right)\, dx &= \lim_{n\to\infty} \left[\frac{3-(-1)}{n} \cdot \sum_{i=1}^{n} f\left(-1 + i \cdot \frac{3-(-1)}{n}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} f\left(-1 + i \cdot \frac{4}{n}\right)\right] = \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left[\left(-1 + i \cdot \frac{4}{n}\right)^2 - 1\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \left[1 - 2i \cdot \frac{4}{n} + i^2 \frac{16}{n^2} - 1\right]\right] = \lim_{n\to\infty} \left[\frac{4}{n} \cdot \sum_{i=1}^{n} \frac{8}{n} \left[\frac{2}{n} i^2 - i\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{32}{n^2} \cdot \sum_{i=1}^{n} \left[\frac{2}{n} i^2 - i\right]\right] = \lim_{n\to\infty} \left[\frac{32}{n^2} \cdot \left[\left(\sum_{i=1}^{n} \frac{2}{n} i^2\right) - \left(\sum_{i=1}^{n} i\right)\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{32}{n^2} \cdot \left[\left(\frac{2}{n} \sum_{i=1}^{n} i^2\right) - \frac{n(n+1)}{2}\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{32}{n^2} \cdot \left[\frac{2}{n} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n^2+n}{2}\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{32}{n^2} \cdot \left[\frac{2n^2+3n+1}{3} - \frac{n^2+n}{2}\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{64}{3} + \frac{32}{n} + \frac{32}{3n^2} - 16 - \frac{16}{n}\right] = \frac{64}{3} + 0 + 0 - \frac{48}{3} - 0 = \frac{16}{3} \quad \square
\end{aligned}$$

2. Use the Right-Hand Rule to compute $\int_{-1}^{3} \left(x^2 - 1\right)\, dx$ using SageMath. *[4]*

You may find SageMath's `sum` command, and perhaps also the `limit` command, to be of use in working through **2**.

Solution. Here is a fairly general code fragment which could be easily modified to compute other definite integrals using the Right-Hand Rule:

[1]:
```
var("n")
var("i")
f = function('f')(x)
f(x) = x^2 - 1
a = -1
b = 3
s = function('s')(n)
s(n) = sum( (b-a)/n * f(a+i*(b-a)/n),i, 1, n )
limit(s(n),n=oo)
```

[1]: 16/3

□

Note. We can check both of our answers above by having SageMath compute the definite integral directly. Continuing the session used to answer **2**:

[2]:
```
integral(f,x,a,b)
```

[2]: 16/3