Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
Trent University, Fall 2021

Quiz #5
Wednesday, 20 October.

Available on Blackboard at 12:00 a.m. Wednesday morning (Eastern Time).
Due on Blackboard by 11:59 p.m. Wednesday night (Eastern Time).
Solutions will be posted on Saturday, 23 October.

Submission: Scanned or photographed solutions are fine, so long as they are legible. Please try to make sure that they are oriented correctly – if they are sideways or upside down, they're rather harder to mark! Submission as a single pdf is strongly preferred, but multiple files and/or other common formats are probably OK in a pinch. Please submit your solutions via Blackboard's Assignments module; if Blackboard does not acknowledge a successful upload, please try again. If uploading to Blackboard fails repeatedly, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

Reminder: Per the course outline, *all work submitted for credit must be written up entirely by yourself, giving due credit to all relevant sources of help and information.* For this and other quizzes, unless stated otherwise, you are permitted to use your textbook and all other course materials, from this and any other mathematics course(s) you have taken or are taking now, but *you may not use any other sources or aids, nor give or receive any help*, except to ask the instructor to clarify questions and to use a calculator (any that you like) to help with your arithmetic and to evaluate functions.

Do both of the following problems.

1. Find the absolute maximum and minimum values, if any, of $f(x) = x^3 - 6x^2 + 9x + 15$ on the interval $[0, 4]$. *[2.5]*

Solution. First, note that $f(x)$ is defined and continuous for all x, which includes the entire closed interval $[0, 4]$. Hence we only need to compare the values of $f(x)$ at the endpoints of the interval and at any critical points in the interval to find the absolute and maximum and minimum values of $f(x)$ on $[0, 4]$.

Second, we look for any critical points $f(x)$ has in the given interval.

$$f'(x) = \frac{d}{dx}\left(x^3 - 6x^2 + 9x + 15\right) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

It follows that $f'(x) = 0$ exactly when $x = 1$ or $x = 3$. Note that both critical points are in the interval $[0, 4]$.

Third, we compare the values of $f(x)$ at the endpoints of the interval and the critical points in the interval:

$$\begin{aligned}
f(0) &= 0^3 - 6 \cdot 0^2 + 9 \cdot 0 + 15 = 0 + 0 + 0 + 15 = 15 \\
f(1) &= 1^3 - 6 \cdot 1^2 + 9 \cdot 1 + 15 = 1 - 6 + 9 + 15 = 19 \\
f(3) &= 3^3 - 6 \cdot 3^2 + 9 \cdot 3 + 15 = 27 - 54 + 27 + 15 = 15 \\
f(4) &= 4^3 - 6 \cdot 4^2 + 9 \cdot 4 + 15 = 64 - 96 + 36 + 15 = 19
\end{aligned}$$

It follows that the maximum value of $f(x) = x^3 - 6x^2 + 9x + 15$ on the interval $[0, 4]$ is 19, which it achieves at the critical point $x = 1$ and the endpoint $x = 4$, and its minimum value on the interval is 15, which it achieves at the endpoint $x = 0$ and the critical point $x = 3$. ■

2. Find the all the relative, as well as the absolute, maximum and minimum values, if any, of $g(x) = xe^{-x^2}$ on its domain. *[2.5]*

SOLUTION. First, observe that $g(x) = xe^{-x^2}$ is defined and continuous for all x, *i.e.* on $(-\infty, \infty)$, so we need to check what happens at any critical points as well as how the function behaves as $x \to -\infty$ and as $x \to \infty$.

Second, we look for the critical points of $g(x)$, with a little help from the Product and Chain Rules.

$$g'(x) = \frac{d}{dx}\left(xe^{-x^2}\right) = 1 \cdot e^{-x^2} + x \cdot e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) = e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = \left(1 - 2x^2\right)e^{-x^2}$$

Since $e^{-x^2} > 0$ for all x, $g'(x) = 0$ exactly when $1 - 2x^2 = 0$, *i.e.* exactly when $x = \pm\frac{1}{\sqrt{2}}$.

Third, we evaluate $g(x)$ at each of the critical points and determine whether they are local maxima or minima (or neither).

$$\begin{aligned}
g\left(\frac{-1}{\sqrt{2}}\right) &= \frac{-1}{\sqrt{2}} \cdot e^{-\left(-1/\sqrt{2}\right)^2} = \frac{-1}{\sqrt{2}} \cdot e^{-1/2} = \frac{-1}{\sqrt{2}} \cdot \frac{1}{\sqrt{e}} = \frac{-1}{\sqrt{2e}} \approx -2.3316 \\
g\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}} \cdot e^{-\left(1/\sqrt{2}\right)^2} = \frac{1}{\sqrt{2}} \cdot e^{-1/2} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{2e}} \approx 2.3316
\end{aligned}$$

We can determine whether these critical points give local maxima, minima, or neither by checking the values of $g(x)$ to either side and between the ccritical points. Since $\pm\frac{1}{\sqrt{2}} \approx \pm 0.7071$, it suffices to evaluate $g(x)$ at $x = -1$, 0, and 1:

$$\begin{aligned}
g(-1) &= -1 \cdot e^{-(-1)^2} = -e^{-1} = -\frac{1}{e} \approx -0.3679 \\
g(0) &= 0 \cdot e^{-0^2} = 0 \cdot 1 = 0 \\
g(1) &= 1 \cdot e^{-1^2} = e^{-1} = \frac{1}{e} \approx 0.3679
\end{aligned}$$

Since $g\left(\frac{-1}{\sqrt{2}}\right) < g(-1)$ and $g\left(\frac{-1}{\sqrt{2}}\right) < g(0)$, $g\left(\frac{-1}{\sqrt{2}}\right) \approx -2.3316$ is a relative minimum value. Similarly, since $g\left(\frac{1}{\sqrt{2}}\right) > g(0)$ and $g\left(\frac{1}{\sqrt{2}}\right) > g(1)$, $g\left(\frac{1}{\sqrt{2}}\right) \approx 2.3316$ is a relative maximum value.

Fourth, we check the behaviour of $g(x)$ as $x \to -\infty$ and as $x \to \infty$, with a little help from l'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to-\infty} g(x) &= \lim_{x\to-\infty} xe^{-x^2} = \lim_{x\to-\infty} \frac{x}{e^{x^2}} \begin{matrix} \to -\infty \\ \to +\infty \end{matrix} = \lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2}} \\
&= \lim_{x\to-\infty} \frac{1}{e^{x^2}\cdot\frac{d}{dx}x^2} = \lim_{x\to-\infty} \frac{1}{2xe^{x^2}} \begin{matrix} \to 1 \\ \to \infty \end{matrix} = 0 \\
\lim_{x\to\infty} g(x) &= \lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} \begin{matrix} \to +\infty \\ \to +\infty \end{matrix} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{e^{x^2}\cdot\frac{d}{dx}x^2} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} \begin{matrix} \to 1 \\ \to \infty \end{matrix} = 0
\end{aligned}$$

Note that since $g(x) = xe^{-x^2} < 0$ when $x < 0$, $g(x)$ approaches 0 from below as $x \to -\infty$. Similarly, since $g(x) = xe^{-x^2} > 0$ when $x > 0$, $g(x)$ approaches 0 from above as $x \to -\infty$.

Finally, we put all of the above together. Since $g\left(\frac{1}{\sqrt{2}}\right) \approx 2.3316$ is a relative maximum, with all values to the left of it being less – note that all values of $g(x)$ left of $x = 0$ are negative – and declining to 0 on the right, $g\left(\frac{1}{\sqrt{2}}\right) \approx 2.3316$ is actually the absolute maximum of $g(x)$ on $(-\infty, \infty)$. Similarly, since $g\left(\frac{-1}{\sqrt{2}}\right) \approx -2.3316$ is a relative minimum, with all values to the right of it being greater – note that all values of $g(x)$ right of $x = 0$ are positive – and dincreasing to 0 on the left, $g\left(\frac{-1}{\sqrt{2}}\right) \approx -2.3316$ is actually the absolute minimum of $g(x)$ on $(-\infty, \infty)$. ■

[Total = 5]